TEXT FLY AND PAGES MISSING WITHIN THE BOOK ONLY

UNIVERSAL
LIBRARY

ELEMENTS

OF

PLANE TRIGONOMETRY

(WITH FIVE-PLACE TABLES)

A TEXT-BOOK FOR HIGH SCHOOLS,
TECHNICAL SCHOOLS AND
COLLEGES

BY

ROBERT E. MORITZ

PH.D. (NEBRASKA), PH.N.D. (STRASSBURG), PROFESSOR OF MATHEMATICS,
UNIVERSITY OF WASHINGTON

NEW YORK
JOHN WILEY & SONS, INC.
LONDON · CHAPMAN & HALL, LIMITED

Printed in U. S. A.

Stanhope Press
F. H. GILSON COMPANY
BOSTON, U.S.A.

PREFACE

TRIGONOMETRY is college mathematics *par excellence*. To at least 90 per cent of all liberal arts students college mathematics means trigonometry and nothing else. It is important, therefore, that the science be presented in as simple and attractive a manner as possible and that it be made more than a mere method of solving triangles. The first the author tries to accomplish by making the treatment less technical than is customary, — by introducing considerable historical matter, — by not presupposing a too ready knowledge of elementary mathematics, and none at all of the topics ordinarily treated in college algebra. To accomplish the second point the angle is made the central idea of the science. This permits the enrichment of the science through the introduction of a variety of concepts and processes ordinarily reserved for advanced courses in mathematics.

Since the treatment departs considerably from that current in textbooks on trigonometry, it is fitting that some of the leading characteristics of the present book should be enumerated at the outset:

First, as to subject-matter:

(1) The book has been planned to cover five months' work at four lessons per week. Each month's work is followed by a set of review exercises. Where less time must be given to the subject, certain advanced chapters may, of course, be omitted.

(2) The introductory chapter on the graphic method of solving triangles is intended to impress the need of a more accurate method, — the method of trigonometry.

(3) A knowledge of logarithms has not been presupposed. For this reason a chapter on logarithms and the use of tables has been incorporated at its proper place. Classes who are properly prepared in logarithms may of course omit this chapter.

(4) Many of the more important results have been derived by two or more independent methods. This has been done,—

(*a*) To give the teacher a choice of methods.

(*b*) To offer the ambitious student the advantage which comes from approaching the same truth from two or more directions.

(*c*) To offer an alternative to the student without a teacher who finds undue difficulty with any one given proof.

(5) It is not intended that all the problems should be assigned to any one class. The problems in each set are carefully graded and arranged as follows,—

(*a*) The first half in each set are very simple applications of the principles and theorems discussed in the preceding sections.

(*b*) The next three or four problems require some originality on the part of the student.

(*c*) The last few problems in each set are for the more ambitious student and frequently give him the opportunity to discover for himself results which are discussed in detail in later sections of the book.

(6) Special care has been bestowed on the applied problems illustrating the solution of right and oblique triangles. In each case there is given first a set of problems involving miscellaneous heights and distances. This is followed by separate sets of applied problems from each of the following sciences: Physics, Engineering, Navigation, Astronomy and Geography, and Elementary Geometry. These lists are probably the most varied and complete that have been published in America in recent years.

(7) Trigonometric curves have received much fuller treatment than is usual. The method of representing functions by curves is developed from first principles. The treatment includes sine curves of given amplitude and wave length, logarithmic and exponential curves, composition of harmonic curves, the catenary, and the curve of damped vibrations.

(8) A special section is devoted to the angle and its functions considered as functions of time.

(9) In developing the theorems of Demoivre and Euler no knowledge of imaginaries is presupposed. The chapter on trigonometric series presupposes no knowledge of series.

(10) The usual inadequate treatment of hyperbolic functions is replaced by a separate chapter in which the analogies between the

circular and hyperbolic functions are developed both analytically and geometrically. This chapter concludes with the determination of the area of a hyperbolic sector.

(11) Abundant historical matter has been introduced throughout the work.

Second, as to method and arrangement:

(12) Logical order has been made subsidiary to order of teaching. To illustrate: the study of the functions of an angle is divided into three parts,—

(*a*) The study of the functions of an acute angle, followed by the solution of right triangles.

(*b*) The study of the functions of the obtuse angle, followed by the solution of oblique triangles.

(*c*) The study of the functions of the general angle, followed by the solution of trigonometric equations.

(13) The discussion of the general angle, of circular measure, and of functions of two or more angles, is postponed until after the solution of the oblique triangle. This makes it possible to complete the subject through the solution of triangles in half a semester, an important consideration for classes in short summer sessions, and for engineering students who begin surveying and trigonometry the same semester. Furthermore it removes the suspicion, so often felt by the student, that the solution of triangles is the sole aim of the science.

(14) Following the plan long since established in Germany and France, only the three principal functions, sine, cosine and tangent, have received detailed treatment. The corresponding results for the reciprocal functions are left as exercises for the student. This plan economizes time and space and leaves the student with a clearer understanding of the entire subject.

(15) Every example worked out in the text is followed by one or more checks. Checks are looked upon as an essential part of every solution. In order to cultivate the use of checks, it would seem best to omit the answers to the exercises, yet every teacher knows the importance of "answers" to the beginner in guiding his first uncertain efforts. In the present text the plan has been adopted of supplying the answers to a part of the exercises only, with explicit directions to the student to check every problem to which no answer is given. This leaves the enforcement of the checking habit largely

with the individual teacher, who may assign as many problems without answers as he deems desirable.

(16) No pains has been spared to impress the student with the limitations in the degree of accuracy, in the answers to problems, imposed by the data, as well as with the limitations in the degree of accuracy due to the use of tables. Superfluous figures and show of accuracy not warranted by the data or the process of computation employed, are nowhere tolerated in the present book.

(17) On the other hand the student is guarded against disregarding figures and remainders without first measuring the effect of the parts neglected on the required results. Thus, in the computation of logarithms by means of the logarithmic series, or of natural functions by means of the sine and cosine series, the effect of the neglected part of each series on the final result has been carefully considered in each case.

The book embodies the author's practical experience of seventeen years in presenting the subject to beginners in colleges and universities. His experience has convinced him that the subject of trigonometry can be so simplified and enriched that it deserves the foremost place in the last year of any high school or the first year of any college curriculum, not only because of its intensely practical value, but chiefly because of its unrivaled cultural value. In the mastery of logarithms, which strips the most complicated and laborious calculations of their difficulties and irksomeness, the student cannot help becoming conscious of the tremendous power of the human mind when properly directed. In the application of algebraic processes and symbols to geometrical and physical magnitudes, he is initiated into a most far-reaching method of modern research, that of analytical geometry. The study of trigonometric curves should give him a working knowledge of an indispensable tool in every field of scientific activity. The study of the trigonometric and logarithmic series, and their use in the computation of logarithmic and natural functions, opens an entirely new field of thought with its importance for practical ends. The actual use of tables familiarizes the student with the principle of interpolation, a knowledge of which is demanded wherever tables are used.

Besides these concepts and processes, the importance of which must appeal to all, there are an abundance of others which open the door to higher realms of thought. The simplest applications of

trigonometry to imaginary and complex numbers reveals a new conception of addition and multiplication; in the determination of the roots of unity an otherwise unsolvable problem is solved in all its generality; imaginary angles lead to the unsuspected region of hyperbolic functions and reveal a new world of symmetry and beauty.

Above all, while contemplating the lifelong self-sacrificing efforts of the master minds who brought the science to its present state of perfection, of men who spent their lives without either pecuniary compensation or popular applause in order to share in the building of the temple of abstract truth, the student must come to a better appreciation of truth for its own sake and be helped in part to a realization of the higher objects of human endeavor.

While writing this book the author has received valuable suggestions from several of his colleagues. Special mention is due Mr. George I. Gavett who supplied some of the applied problems from engineering, and to Mr. Allen Carpenter who read the entire manuscript and verified many of the answers to the problems.

GREEK ALPHABET

α	pronounced	*alpha.*	ν	pronounced	*nu.*
β	"	*beta.*	o	"	*omikron.*
γ	"	*gamma.*	π	"	*pi.*
δ	"	*delta.*	ρ	"	*rho.*
ϵ	"	*epsilon.*	σ	"	*sigma.*
η	"	*eta.*	τ	"	*tau.*
ζ	"	*zeta.*	υ	"	*upsilon.*
θ	"	*theta.*	ξ	"	*xi*
ι	"	*iota.*	ψ	"	*psi.*
κ	"	*kappa.*	ϕ	"	*phi.*
λ	"	*lambda.*	χ	"	*chi.*
μ	"	*mu.*	ω	"	*omega*

CONTENTS

CHAPTER I

INTRODUCTION

CHAPTER II

TRIGONOMETRIC FUNCTIONS OF AN ACUTE ANGLE

CHAPTER III

SOLUTION OF RIGHT TRIANGLES BY NATURAL FUNCTIONS

CHAPTER IV

LOGARITHMS

CHAPTER V

Logarithmic Solution of Right Triangles

CHAPTER VI

Functions of an Obtuse Angle

CHAPTER VII

Properties of Triangles

CHAPTER VIII

Solution of Oblique Triangles

CHAPTER IX

The General Angle and its Measures

CHAPTER X

Functions of any Angle

CHAPTER XI

Functions of Two or More Angles

CHAPTER XII

Trigonometric Equations

CHAPTER XIII

Trigonometric Curves

CHAPTER XIV

Trigonometric Representation of Complex Numbers

CHAPTER XV

Trigonometric Series and Calculation of Tables

CHAPTER XVI

Hyperbolic Functions

PLANE TRIGONOMETRY

CHAPTER I

INTRODUCTION

In order to work the exercises in this chapter the student should be provided with a pair of compasses, a protractor, and a graduated ruler divided into tenths of a unit.

1. Graphic Solution of Triangles. In plane geometry it is shown that the six parts (three sides and three angles) of any plane triangle are so related that any three parts suffice to determine the shape of the triangle, and if one of the known parts is a side, the size of the triangle is also determined. Furthermore it is shown how to construct the triangle when a sufficient number of parts is given. All possible cases come under one or another of the following four cases.

To construct the triangle when there is given,—

I. One side and two angles.
II. Two sides and an angle opposite one of them.
III. Two sides and the included angle.
IV. Three sides.

Usually we have given not the actual lines and angles but their measures. From these measures lines and angles corresponding to the actual lines and angles may then be constructed by means of suitable instruments. Such instruments are,—

1. A *graduated straight-edge* for the construction and measurement of straight lines of definite lengths. The smallest divisions of the straight-edge should be decimal, either millimeters or tenths of an inch.

2. A *pair of compasses* for the construction of circles and circular arcs.

3. A *protractor* for the construction and measurement of plane angles of definite magnitudes.

EXAMPLE 1. It is required to construct a triangle which has two sides equal to 2.5 inches and 1.75 inches respectively and the included angle equal to 36°.

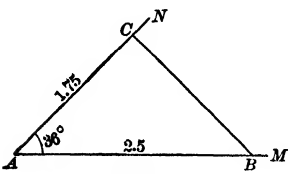


Fig. 1.

Solution. By means of the protractor construct an angle MAN (Fig. 1) equal to 36°. On AM measure off AB equal to 2.5 inches. On AN measure off AC equal to 1.75 inches. Join B and C by a straight line. ABC is the required triangle.

The numerical values of the parts which were not known at the outset may now be found by measurement. BC is thus found to be 1.49 inches, and by means of the protractor, angles B and C are found to be approximately 43.5° and 100.5° respectively.

If it is not possible or convenient to construct the triangle full size, a similar triangle may be constructed on a *reduced scale;* that is, any unit or a fraction of a unit on the scale may be taken to represent any unit occurring in the problem. Thus lines 3 and 4 inches long may be employed in the solution of a triangle whose sides are 30 and 40 miles respectively. The angles of the reduced triangle will of course be equal to the angles of the triangle represented.

Similarly, the unknown parts of a triangle which is too small for actual construction, say some microscopic triangle, may be found by measurement from a similar triangle drawn on an *enlarged scale.*

EXAMPLE 2. One side of a triangle measures 600 miles, and the angles adjacent to this side measure 23° and 100° respectively. Find the remaining parts of the triangle.

Solution. Let $\frac{1}{2}$ inch represent 100 miles. Then a line AB drawn 3 inches long will represent 600 miles. At A and B draw the angles BAC and ABD 23° and 100° respectively. Let BD intersect AC at E. ABE will represent the required triangle.

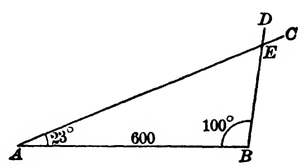


Fig. 2.

Angle E measures 57°, which of course could have been found otherwise by subtracting the sum of the angles A and B from 180°. AE and BE are found to measure approximately 3.52 and 1.40 inches respectively. Remembering that each $\frac{1}{2}$ inch represents 100

miles, the actual lengths represented by AE and BE are approximately 704 and 280 miles respectively.

Solutions, like the foregoing, in which geometrical drawings to a scale are employed instead of numerical calculations, are called *graphic solutions.*

Exercise 1

1. Review the following propositions in geometry. A, B, C represent the three angles of any triangle and a, b, c the sides opposite these angles.

   a. Given A, B, c; to construct the triangle.
   b. Given a, b, C; to construct the triangle.
   c. Given a, b, c; to construct the triangle.
   d. Given a, b, A; to construct the triangle.
   e. Under what conditions will (d) give rise to two different solutions? To only one solution?

The following problems are to be solved by the graphic method.*

2. Given $a = 5, b = 4, c = 7$; find the angles to the nearest 15′.
Ans. $A = 44° 30', B = 34°, C = 101° 30'$.

3. Given $b = 4, c = 5, C = 90°$; find the third side and the angles to the nearest 15′. *Ans.* $a = 3, A = 37°, B = 53°$.

4. Given $b = 270, c = 600, A = 100°$; find the third side correct to the nearest integer. *Ans.* $a = 700$.

5. Given $a = 0.029, B = 32° 15', C = 136° 45'$; find the remaining sides. *Ans.* $b = 0.081, c = 0.104$.

6. Given $a = 42, b = 51, A = 55°$; find the approximate measures of the remaining parts. *Ans.* $c = 33.6, B = 84°, C = 41°$;
or $c = 24.9, B = 96°, C = 29°$.

7. Given $A = 44° 30', B = 57°, C = 78° 30'$; find the ratios between the sides opposite these angles.
Ans. Approximately $a : b : c = 5 : 6 : 7$.

* In order to employ the graphic method successfully the student must practice accuracy. Two pencils of medium hardness should be used, one sharpened to a point for marking distances, the other sharpened like a chisel for drawing lines. The pencil points are easily kept sharp with the aid of a piece of fine sandpaper. The lines should be drawn sharply and they should bisect the points through which they are intended to pass. In measuring the required parts, beginners should estimate angles to quarters of a degree and lengths to quarters of the smallest division of the scale.

2. Solution of Practical Problems by the Graphic Method. Many important practical problems, in which a high degree of accuracy is not essential, can be easily solved by the graphic method. Suppose it is required to find the approximate distance AB across a lake or swamp, without actually measuring it. This may be accomplished in various ways, one of which is as follows:

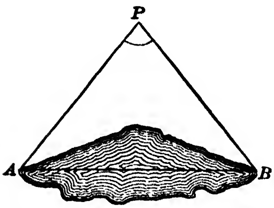


Fig. 3.

Select some point P from which both A and B are visible, and measure the distances AP and BP and also the angle APB.* This gives two sides and the included angle of the triangle APB from which AB may be found by the method of the preceding article.

Similarly the heights of towers and trees and mountains, of clouds and shooting stars, the distances through impenetrable forests across swamps and through mountains, the widths of rivers, ravines and canyons, may be determined. Even the distances between celestial objects may be approximated by the graphic method after certain other distances and angles have been measured.

EXAMPLE 1. In order to determine the width of a river, the distance between two points A and B close to the bank of the river was measured and was found to be 600 feet. The angles BAP and ABP, formed with a point P close to the opposite bank of the river, were also measured and were found to be 50° and 36° respectively. Required the approximate width of river.

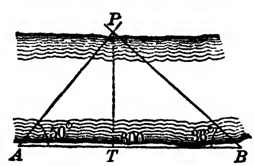


Fig. 4.

Solution. Select a suitable scale, say 1 inch to 100 feet, and construct a triangle ABP, having $AB = 6$ inches and the adjacent angles equal to 50° and 36° respectively. From P draw PT perpendicular to AB. PT will represent the width of the river. Measure PT. PT will be found to measure 2.7 inches, and since each inch represents 100 feet, the width of the river is 270 feet.

DEFINITIONS. Let P be any point and O the position of the observer. Through P draw a vertical line, and through O draw a horizontal line meeting the vertical line in H.

* The angle between two visible objects is readily measured by means of an instrument called a transit.

If P is above H, as in the upper figure, the angle HOP is called the *angle of elevation* of the point P as seen from O,

If P is below H, as in the lower figure, the angle HOP is called the *angle of depression* of the point P as seen from O.

It is obvious that the angle of elevation or of depression of an object depends upon the position of the observer.

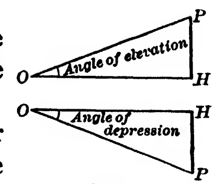


Fig. 5.

EXAMPLE 2. From a point P at the foot of a mountain, the angle of elevation of the summit M is measured and is found to be 30°; after walking two miles toward the summit on an incline averaging 15°, the angle of elevation is found to measure 45°. Required the height of the mountain.

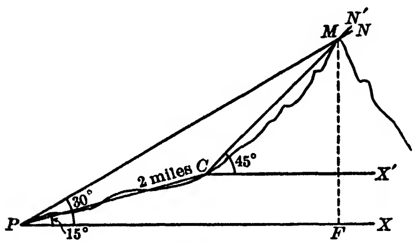


Fig. 6.

Solution. Draw a horizontal line PX. Construct an angle $XPN = 30°$; then PN represents the direction in which the summit of the mountain is seen from P. Construct angle $XPC = 15°$, and take PC two units in length. Then, if each unit represents one mile, C will represent the position from which the second observation was made.

Through C draw CX' parallel to PX, and construct an angle $X'CN' = 45°$. Then CN' represents the direction in which the summit is seen from C.

Since the summit is on each of the lines PN and CN', it must be located at their point of intersection M.

Draw MF perpendicular to PX. Then MF will represent the height of the mountain on the same scale on which PC represents two miles. Measure MF. If PC was taken equal to 6 inches, MF will measure 5.8 inches. Since 3 inches represents one mile, MF represents 1.933 miles, or 10,200 feet approximately.

EXERCISE 2

The following problems are to be solved graphically. The student is expected to obtain distances correct to three figures and angles correct to nearest 15′.

1. At a distance of 400 feet from the foot of a tree, the top of the tree subtends an angle of 20°. Find the height of the tree.
Ans. 145.6 ft.

2. A straight road leads from a town A to a town B 8 miles distant; another road leads from A to a third town C 10 miles distant. The angle between the roads is 65°. How far is it from B to C?
Ans. 9.82 mi.

3. What is the altitude (= angle of elevation) of the sun, when a building 75 feet high casts a shadow 190 feet long on a horizontal plane? *Ans.* 21° 30′.

4. The great pyramid of Gizeh is 762 feet square at its base and each face makes an angle of 51° 51′ with the horizontal plane. Determine the height of the pyramid, assuming that it comes to an apex. *Ans.* 485 feet.

5. As a matter of fact, the pyramid mentioned in Problem 4 does not come to a point, but terminates in a platform 32 feet square. Find the actual height of the pyramid. *Ans.* 465 feet.

6. An observer on board ship sees two headlands in a straight line N. 35° E. The ship sails northwest for 5 miles, when one of the headlands appears due east and the other due northeast. How far apart are the headlands?

7. Two observers on opposite sides of a balloon observe the balloon at the same instant and find its angles of elevation to be 56° and 42° respectively. The observers are one mile apart. Find the height of the balloon at the time the observations were taken.
Ans. 0.6 mi. nearly.

8. In order to determine the distance across a swamp, a distance AB was laid off 100 yards long, and at each extremity of the line AB the angles were measured between the other extremity of the line and each of two stakes P and Q placed at opposite ends of the swamp. At one extremity of the line the angles measured 35° and 85° respectively, at the other end the angles measured 40° and 121° respectively. Find the distance PQ.

9. Find the perimeter of a regular polygon of 7 sides inscribed in a circle whose radius is 10 feet. *Ans.* 60.75 ft.

10. The sides of a triangle are $a = 10$, $b = 12$, $c = 15$ respectively. Find the radii of the inscribed and of the circumscribed circles and the angles of the triangle. *Ans.* $r = 3.23$, $R = 7.52$,
$A = 41° 30′$, $B = 53°$, $C = 85° 30′$.

3. Inadequacy of the Graphic Method. The graphic method of solving triangles, though exceedingly simple and useful, is not sufficiently accurate for all purposes. For instance, in the last problem of Exercise 2 the results obtained by the graphic method are:

$$r = 3.23, \quad R = 7.52, \quad A = 41^\circ 30', \quad B = 53^\circ, \quad C = 85^\circ 30',$$

while the more accurate results, obtained by a method to be described later, are:

$$r = 3.2331, \quad R = 7.5236, \quad A = 41^\circ 38' 59'',$$
$$B = 52^\circ 53' 27'', \quad C = 85^\circ 27' 34''.$$

The causes of the inaccuracies are manifold. First of all the divisions of the scale used in measuring are not indefinitely small; if they were the eye could not distinguish them. Besides this the graphic method is subject to many other unavoidable errors. The instruments employed in the construction of lines and angles are imperfect. The straight-edge is not perfectly straight, the divisions of the scale are not exactly equidistant. The points used in the construction are not true points but dots having dimensions; likewise the lines drawn are not true lines but pencil or pen marks of unequal widths. Again, neither the hand which draws the pencil nor the eye which guides it is perfectly steady, and so on. All these sources of error are unavoidable. By employing better instruments and by using greater care, these errors may be diminished, but they cannot be entirely eliminated.

4. Definition of Trigonometry. There exists another method of solving triangles which is free from all the errors above mentioned and enables us to obtain results correct to any desired degree of accuracy. This method consists in the *computation* of the unknown parts of a triangle from the numerical values of the given parts. The development of this method has given rise to a separate branch of mathematics, called trigonometry.* Trigonometry considers the properties of angles and certain ratios associated with angles, and

* The word Trigonometry comes from two Greek words, *trigonon* = triangle, and *metron* = measure. The method was originated in the second century B.C. by Hipparchus and other early Greek astronomers in their attempts to solve certain spherical triangles. The term trigonometry was not used until the close of the sixteenth century.

applies the knowledge of these properties to the solution of triangles and various other algebraic and geometric problems. Incidentally trigonometry considers also certain time-saving aids in computation such as logarithms, which are generally employed in the solution of triangles. Briefly stated,—

Trigonometry is the science of angular magnitudes and the art of applying the principles of this science to the solution of problems.

CHAPTER II

TRIGONOMETRIC FUNCTIONS OF AN ACUTE ANGLE

5. Definition of Function. *When two variables are so related that the value of the one depends upon the value of the other, the one is said to be a function of the other.*

EXAMPLES. The area of a square is a function of its side. The volume of a sphere is a function of its radius. The velocity of a falling body is a function of the time elapsed since it began to fall. The output of a factory is a function of the number of men employed. In the expression $y = \frac{(x-1)}{(x+1)}$, y depends upon x for its value, hence y is a function of x. Similarly $x^2 - 1$, $x^3 + x - 3$, $ax + \frac{b}{x} - c$, are functions of x. $t^2 - 3t$ is a function of t, etc.

6. Definition of Reciprocal. *If the product of two quantities equals unity, each is said to be the reciprocal of the other.*

For example, if $xy = 1$, x is the reciprocal of y, and y is the reciprocal of x. $\frac{1}{2}$ is the reciprocal of 2, and 2 is the reciprocal of $\frac{1}{2}$, for $\frac{1}{2} \times 2 = 1$. In general, $\frac{a}{b}$ and $\frac{b}{a}$ are reciprocals since $\frac{a}{b} \cdot \frac{b}{a} = 1$. From $xy = 1$ it follows that $x = \frac{1}{y}$, and $y = \frac{1}{x}$, that is,—

The reciprocal of any quantity is unity divided by that quantity.

7. The Six Trigonometric Functions of an Acute Angle. Let A be any acute angle, B any point on either side of the angle, and ABC the right triangle formed by drawing a perpendicular from B to the other side of the angle. Denote AC, the side adjacent to the angle A, by b (for base), BC, the side opposite the angle A, by a (for altitude), and the hypotenuse AB by h.

The three sides of the right triangle form six different ratios, namely,

$$\frac{a}{h}, \quad \frac{b}{h}, \quad \frac{a}{b},$$

and their reciprocals

$$\frac{h}{a}, \quad \frac{h}{b}, \quad \frac{b}{a}.$$

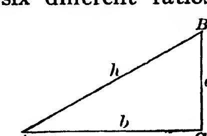


Fig. 7.

a. These ratios do not depend upon the distance of the point B from the vertex of the angle; that is, each of the six ratios will have the same value for every other point B' located on either one of the sides of the angle.

For if B' be any other point on AB or AB produced, or on AC or AC produced, and the perpendicular $B'C'$ be drawn to the other side, the triangle $AB'C'$ will be similar to triangle ABC and therefore

$$\frac{B'C'}{AB'} = \frac{BC}{AB} = \frac{a}{h},$$

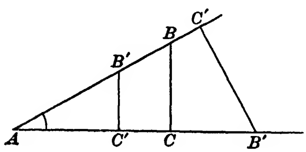


Fig. 8.

and similarly for each of the other ratios.

b. The ratios differ for different angles, for if the ratios were equal, the corresponding triangles would be similar (why?), and corresponding angles equal, which is contrary to the hypothesis that they are different.

Since the ratios depend upon the angle for their values, they are functions of the angle according to the general definition of a function given in 5. Each of these functions has received a special name. Referring to Fig. 7,

$\frac{a}{h}$, that is, $\frac{\textit{side opposite angle } A}{\textit{hypotenuse}}$, is called the **sine** of angle A;

$\frac{b}{h}$, that is, $\frac{\textit{side adjacent angle } A}{\textit{hypotenuse}}$, is called the **cosine** of angle A;

$\frac{a}{b}$, that is, $\frac{\textit{side opposite angle } A}{\textit{side adjacent angle } A}$, is called the **tangent** of angle A;

$\frac{h}{a}$, the *reciprocal of the sine*, is called the **cosecant** of angle A;

$\frac{h}{b}$, the *reciprocal of the cosine*, is called the **secant** of angle A;

$\frac{b}{a}$, the *reciprocal of the tangent*, is called the **cotangent** of angle A.

The six functions just defined are variously known as the *trigonometric*, *circular*, or *goniometric* functions: trigonometric, because they form the basis of the science of trigonometry; circular, because of their relations to the arc of a circle, as will be shown presently; goniometric, because of their use in determining angles, from *gonia*, a Greek word meaning angle. The expressions sine of angle A,

cosine of angle A, etc., are abbreviated to sin A, cos A, tan A, csc A or cosec A, sec A, cot A.

Besides these six functions, two others are sometimes used,* viz.,

versed sine $A = 1 -$ cosine A, abbreviated to **vers** A;

coversed sine $A = 1 -$ sine A, abbreviated to **covers** A.

The definitions of the first six trigonometric functions must be thoroughly memorized. The first three are especially important and should be memorized in the following form:

Given an acute angle in a right triangle, —

The **sine** *of the angle is the ratio of the side opposite the angle to the hypotenuse.*

The **cosine** *of the angle is the ratio of the side adjacent the angle to the hypotenuse.*

The **tangent** *of the angle is the ratio of the side opposite the angle to the adjacent side.*

The remaining three functions may be remembered most readily by the aid of the reciprocal relations, —

$$\sin A \cdot \operatorname{cosec} A = 1,$$
$$\cos A \cdot \sec A = 1,$$
$$\tan A \cdot \cot A = 1,$$

that is,

$$\operatorname{cosec} A = \frac{1}{\sin A}, \qquad \sec A = \frac{1}{\cos A}, \qquad \cot A = \frac{1}{\tan A}.$$

It will aid the memory to observe that only one **o** appears in each pair of reciprocal functions.

It should be noticed that while a, b, and h are lines, the ratio of any two of them is an abstract number; that is, the trigonometric functions are abstract numbers. Again, the expressions sin A, cos A, tan A, etc., are single symbols which cannot be separated. sin has no meaning except as it is associated with some angle, just as the symbol $\sqrt{\ }$ has no meaning except when used in connection with some quantity, as in $\sqrt{a}$, $\sqrt{4}$, etc.

EXAMPLE 1. The sides of a right triangle are 3, 4, 5. Find all the trigonometric functions of the angle A opposite the side 3.

* Especially in navigation.

Solution. The hypotenuse of the triangle equals 5. Hence, applying the definitions, we have

$$\sin A = \frac{\text{side opposite}}{\text{hypotenuse}} = \tfrac{3}{5}.$$

Similarly, $\cos A = \tfrac{4}{5}$, $\tan A = \tfrac{3}{4}$.

The cotangent of A is the reciprocal of the tangent, hence

$$\cot A = 1 \div \tfrac{3}{4} = \tfrac{4}{3}, \text{ similarly, } \sec A = 1 \div \tfrac{4}{5} = \tfrac{5}{4}, \csc A = \tfrac{5}{3}.$$

Example 2. The legs of a right triangle are a and b; find the functions of the angle A opposite the side a.

Solution. The hypotenuse $h = \sqrt{a^2 + b^2}$; hence

$$\sin A = \frac{a}{h} = \frac{a}{\sqrt{a^2+b^2}}, \quad \cos A = \frac{b}{h} = \frac{b}{\sqrt{a^2+b^2}}, \quad \tan A = \frac{a}{b},$$

$$\csc A = \frac{h}{a} = \frac{\sqrt{a^2+b^2}}{a}, \quad \sec A = \frac{h}{b} = \frac{\sqrt{a^2+b^2}}{b}, \quad \cot A = \frac{b}{a}.$$

Exercise 3

1. A right triangle has its sides equal to 5, 12, 13. Calculate the six functions of the angle A opposite the side 5.

Ans. $\sin A = \tfrac{5}{13}$, $\cos A = \tfrac{12}{13}$, $\tan A = \tfrac{5}{12}$,
$\csc A = \tfrac{13}{5}$, $\sec A = \tfrac{13}{12}$, $\cot A = \tfrac{12}{5}$.

2. In the same triangle calculate the functions of the angle B opposite the side 12.

Ans. $\sin B = \tfrac{12}{13}$, $\cos B = \tfrac{5}{13}$, $\tan B = \tfrac{12}{5}$,
$\csc B = \tfrac{13}{12}$, $\sec B = \tfrac{13}{5}$, $\cot B = \tfrac{5}{12}$.

3. By comparing the answers in problem 2 with those in problem 1, and remembering that $A + B = 90°$, write down six equations, of which the following is the first: $\sin A = \cos B = \cos (90° - A)$.

Ans. $\cos A = \sin B = \sin (90° - A)$,
$\tan A = \cot B = \cot (90° - A)$,
$\sec A = \csc B = \csc (90° - A)$, etc.

4. Show that for any acute angle the following equations are true:

$$\sin (90° - A) = \cos A, \qquad \sec (90° - A) = \csc A,$$
$$\cos (90° - A) = \sin A, \qquad \csc (90° - A) = \sec A,$$
$$\tan (90° - A) = \cot A, \qquad \cot (90° - A) = \tan A.$$

(Suggestion. Consider A and $90° - A$ the two acute angles of a right triangle, and express the functions of each angle in terms of the sides of the triangle.)

5. The two legs of a right triangle are 8 and 15. Write down the functions of the angle A opposite the side 8; also the functions of the angle B adjacent the side 8.

Ans. $\sin A = \frac{8}{17}$, $\cos A = \frac{15}{17}$, $\tan A = \frac{8}{15}$,
$\sin B = \frac{15}{17}$, $\cos B = \frac{8}{17}$, $\tan B = \frac{15}{8}$.

6. By using the results of problem 5, show that

$$\frac{\sin A}{\cos A} = \tan A \text{ and } \frac{\cos A}{\sin A} = \cot A.$$

7. One side of a right triangle is 9 and the hypotenuse is 41. Compute the functions of the angle A included between the hypotenuse and the given side.

Ans. $\sin A = \frac{40}{41}$, $\cos A = \frac{9}{41}$, $\tan A = \frac{40}{9}$,
$\csc A = \frac{41}{40}$, $\sec A = \frac{41}{9}$, $\cot A = \frac{9}{40}$.

8. Two legs of a right triangle are $p^2 - q^2$, and $2\,pq$ respectively. Find the sine, cosine and tangent of the angle B opposite the side $2\,pq$.

Ans. $\sin B = \dfrac{2\,pq}{p^2 + q^2}$, $\cos B = \dfrac{p^2 - q^2}{p^2 + q^2}$, $\tan B = \dfrac{2\,pq}{p^2 - q^2}$.

9. Given $\sin A = \frac{4}{5}$, find vers A and covers A.

Ans. vers $A = \frac{2}{5}$, covers $A = \frac{1}{5}$.

EXERCISE 4

1. Compute all the functions of 45°.

Ans. $\sin 45° = \cos 45° = \frac{1}{2}\sqrt{2} = 0.707$, $\tan 45° = 1$,
$\csc 45° = \sec 45° = \sqrt{2} = 1.414$, $\cot 45° = 1$.

(Suggestion. Construct a right triangle having an angle 45°, and denote each of the equal sides by a.)

2. Compute all the functions of 30°.

Ans. $\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{1}{2}\sqrt{3}$, $\tan 30° = \frac{1}{3}\sqrt{3}$,
$\csc 30° = 2$, $\sec 30° = \frac{2}{3}\sqrt{3}$, $\cot 30° = \sqrt{3}$.

(Suggestion. If one angle of a right triangle is double the other, then the hypotenuse is double the shorter side. Call the shorter side a.)

3. By measurement find the functions of 15°.

Ans. $\sin 15° = 0.26$, $\cos 15° = 0.97$, $\tan 15° = 0.28$,
$\csc 15° = 3.87$, $\sec 15° = 1.03$, $\cot 15° = 3.73$.

(Suggestion. By means of a protractor, or otherwise, construct an

angle of 15°. Draw a perpendicular to one side forming a right triangle. Measure the sides of the triangle, and compute the ratios.)

4. Given $\sin A = \frac{5}{7}$, construct and measure the angle.

Ans. $A = 45^\circ\ 35'$.

5. Given $\cos A = \frac{2}{3}$, construct and measure the angle.

Ans. $A = 48^\circ\ 11'$.

6. Given $\tan A = \frac{6}{5}$, construct and measure the angle.

Ans. $A = 50^\circ\ 12'$.

7. Show that

$$\frac{\sin A}{\cos A} = \tan A, \quad \frac{\cos A}{\sin A} = \cot A,$$

A being any acute angle.

8. Show that

$$\sin^2 A + \cos^2 A = 1,^*$$
$$\tan^2 A + 1 = \sec^2 A,$$
$$\cot^2 A + 1 = \csc^2 A.$$

(Suggestion. Remember that $a^2 + b^2 = h^2$.)

9. Given $\sin A = \frac{7}{25}$, find all the other functions.

Ans. $\sin A = \frac{7}{25}$, $\cos A = \frac{24}{25}$, $\tan A = \frac{7}{24}$,
$\csc A = \frac{25}{7}$, $\sec A = \frac{25}{24}$, $\cot A = \frac{24}{7}$.

10. Show that as an angle increases from 0° toward 90°, its sine, tangent and secant increase, while its cosine, cotangent and cosecant decrease.

11. Show that every sine and cosine is a proper fraction, while the tangent and cotangent may have any value large or small.

8. Trigonometric Functions Determined Approximately by Measurement. There are various ways of computing the trigonometric functions of a given angle. The results of such computations for the sines, cosines, tangents and cotangents of angles between 0° and 90° have been put together into tables, known as *tables of natural functions.* We shall learn how to use such tables and later how to calculate them. It is of value to the beginner to know how approximate values of the functions may be obtained graphically.

* $\sin^2 A$ means $(\sin A)^2$, $\tan^2 A$ means $(\tan A)^2$, etc., and generally $\sin^n A = (\sin A)^n$, $\tan^n A = (\tan A)^n$, etc., except when $n = -1$. The meaning of $\sin^{-1} A$, $\tan^{-1} A$, etc., will be explained later.

EXAMPLE 1. Find graphically the functions of 25°.

Solution. By means of a protractor construct an angle MON = 25°. Take OB any convenient length, say 10 inches, and from O as a center and with OB as a radius describe an arc BC cutting ON in C. Draw CE and BA perpendicular to OB. By measurement we find $CE = 4.23$ inches, $OE = 9.06$ inches, $AB = 4.66$ inches, and by construction $OB = OC = 10$ inches. Hence

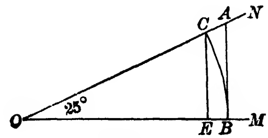


Fig. 9.

$$\sin 25^\circ = \frac{EC}{OC} = \frac{4.23}{10} = 0.423,\ \csc 25^\circ = \frac{1}{\sin 25} = 2.364,$$

$$\cos 25^\circ = \frac{OE}{OC} = \frac{9.06}{10} = 0.906,\ \sec 25^\circ = \frac{1}{\cos 25} = 1.104,$$

$$\tan 25^\circ = \frac{BA}{OB} = \frac{4.66}{10} = 0.466,\ \cot 25^\circ = \frac{1}{\tan 25} = 2.145.$$

Observe that we have used two different triangles, the triangle COE to obtain the sine, cosine and their reciprocals, and a second triangle AOB to obtain the tangent and cotangent. This was done in order to have in each case 10 for a divisor. If for instance we had used the triangle AOB only, we would have had

$$\sin 25^\circ = \frac{BA}{OA} = \frac{4.66}{11.03} = 0.423, \text{ instead of } \frac{4.23}{10} \text{ as above.}$$

EXERCISE 5

1. Obtain by measurement the sine, cosine and tangent of 40°.

Ans. $\sin 40^\circ = 0.643$, $\cos 40^\circ = 0.766$, $\tan 40^\circ = 0.839$.

2. Find the sine, cosine and tangent of 35°. To avoid constructing the triangle and measuring the necessary lines, we may make use of the diagram facing page 16. $\sin 35^\circ = \dfrac{AB}{OA}$. Now $OA = OR = 100$, and the measure of $AB = 57.4$ may be read off on the vertical scale. Hence $\sin 35^\circ = \dfrac{57.4}{100} = 0.574$.

Similarly,

$$\cos 35^\circ = \frac{OB}{OA} = \frac{81.9}{100} = 0.819.$$

To find the tangent it is better to use the triangle OPR, from which

$$\tan 35^\circ = \frac{RP}{OR} = \frac{70}{100} = 0.70.$$

3. With the aid of Fig. 10 find each of the results given in the following table.

(Suggestion. Choose your triangle so that the denominator of the fraction equals 100 or some other integer.)

NATURAL FUNCTIONS.

Angle.	sin	cos	tan	cot	
5°	0.087	0.996	0.087	11.430	85°
10°	0.174	0.985	0.174	5.671	80°
15°	0.259	0.966	0.268	3.732	75°
20°	0.342	0.940	0.364	2.747	70°
25°	0.423	0.906	0.466	2.145	65°
30°	0.500	0.866	0.577	1.732	60°
35°	0.574	0.819	0.700	1.428	55°
40°	0.643	0.766	0.839	1.192	50°
45°	0.707	0.707	1.000	1.000	45°
	cos	sin	cot	tan	Angle

Explanation of table. For angles in the left-hand column the names of the functions appear on top, for angles in the right-hand column the names of the functions appear at the bottom. Thus the number 0.423, which is in the column headed "sin" and has 25° to the left of it, is the sine of 25°; the same number being in the column which has "cos" at the bottom, and having 65° opposite it in the right column, is also the cosine of 65°.

4. By use of the table, express the numbers 0.174, 0.866, 0.643, 0.707, both as sines and cosines.

Ans. $0.174 = \sin 10^\circ = \cos 80^\circ$, $0.866 = \sin 60^\circ = \cos 30^\circ$, etc.

5. By use of the table, express the numbers 0.364, 2.145 and 1.000 both as tangents and cotangents.

6. Every number in the second and third columns is the sine of one angle and the cosine of another angle. What relation do you observe between each pair of angles?

Every number in the fourth and fifth columns is the tangent of one angle and the cotangent of another. What relation exists between each pair of angles?

7. By examining the table, verify the following statements and, if you can, give reasons for them.

a. Every sine and cosine is a fraction less than unity.

b. Every tangent of an angle less than 45° is a fraction less than 1.

c. Every tangent of an angle greater than 45° is some number greater than unity.

d. The sine and tangent of an angle increase as the angle increases.

e. The cosine and cotangent of an angle decrease as the angle increases.

f. The sine and tangent of a small angle are nearly equal.

9. Given One of Its Functions, To Construct the Angle. In the last article it was shown how to find by measurement the functions of a given angle. We will now consider the converse problem, that is, how to construct the angle when one of its functions is given.

EXAMPLE 1. To construct an angle whose tangent is $\frac{3}{4}$.

Solution. Take AC 4 units in length and at C construct a perpendicular CB 3 units in length. Join A and B. Then CAB is the required angle, for $\tan CAB = \dfrac{CB}{AC} = \frac{3}{4}$.

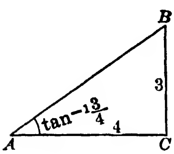


Fig. 11.

The expression

A is the angle whose tangent is $\frac{3}{4}$

is written in short

$$A = \tan^{-1}\tfrac{3}{4},$$

and is read in either of three ways:

a. A is the angle whose tangent is $\frac{3}{4}$.

b. A is the inverse tangent $\frac{3}{4}$.

c. A is the arctangent $\frac{3}{4}$.

Similarly, if

$$y = \sin x$$

then

$$x = \sin^{-1} y,$$

in words, if y equals the sine of x,

then x equals the angle whose sine is y,

or x equals the inverse sine y,

or x equals the arcsine y.

Corresponding meanings are given to the symbols

$$y = \cos^{-1}x,\ y = \csc^{-1}x,\ y = \sec^{-1}x,\ y = \cot^{-1}x.$$

EXAMPLE 2. To construct an angle whose sine is $\frac{4}{7}$.

Solution. At a point C in a line CD of indefinite length, construct a perpendicular CB 4 units in length. From B as a center, with a radius 7 units in length, draw an arc cutting CD at A. Join A and B. Then the angle CAB is the required angle, for

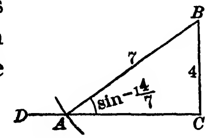


Fig. 12.

$$\sin CAB = \frac{CB}{AB} = \frac{4}{7}.$$

Another way of constructing $\sin^{-1}\frac{4}{7}$ is to construct a semicircle having a diameter AB 7 units long. From B as a center, with a radius 4 units long, describe an arc cutting the semicircle at C. Join A and C. Then BAC is the required angle. (Draw the figure and give reasons.)

EXAMPLE 3. In the preceding example, find each of the other functions of the angle A.

Solution. Two sides of the right triangle ABC being known, the third side is easily found.

$$AC = \sqrt{AB^2 - BC^2} = \sqrt{7^2 - 4^2} = \sqrt{33}.$$

Hence

$$\cos A = \frac{AC}{AB} = \frac{\sqrt{33}}{7} = 0.821, \quad \tan A = \frac{BC}{AC} = \frac{4}{\sqrt{33}} = 0.696,$$

$$\sec A = \frac{1}{\cos A} = \frac{7}{\sqrt{33}} = 1.218, \quad \cot A = \frac{1}{\tan A} = \frac{\sqrt{33}}{4} = 1.436,$$

$$\csc A = \frac{1}{\sin A} = \frac{7}{4} = 1.750.$$

Observe that the expressions, —

$$\sin^{-1}\frac{4}{7},\ \cos^{-1}\frac{\sqrt{33}}{7},\ \tan^{-1}\frac{4}{\sqrt{33}},\ \csc^{-1}\frac{7}{4},\ \sec^{-1}\frac{7}{\sqrt{33}},\ \cot^{-1}\frac{\sqrt{33}}{4},$$

all represent the same angle.

EXERCISE 6

Construct an acute angle equal to A, when,—

1. $\sin A = \frac{2}{3}$.
2. $\tan A = \frac{5}{2}$.
3. $\cos A = 0.5$.
4. $\sec A = \frac{5}{3}$.
5. $\tan A = 4$.
6. $\cos A = 0.45$.

7. $\cos A = \frac{m}{n}$. 8. $\sin A = k$. 9. $\tan A = \frac{1}{k}$.

10. Read in three different ways the expressions,—

$$A = \sin^{-1}\tfrac{2}{3}, \quad B = \tan^{-1} 3, \quad x = \csc^{-1} 3.5, \quad y = \cot^{-1}\sqrt{3}.$$

11. Construct and measure in degrees each of the following angles,—

$$A = \sin^{-1}\tfrac{1}{3}, \quad B = \cos^{-1}\tfrac{1}{3}, \quad C = \tan^{-1}\tfrac{1}{3}, \quad D = \cot^{-1}\tfrac{1}{3}.$$

Ans. $A = 19.5°$; $B = 70.5°$; $C = 18.5°$; $D = 71.5°$; approximately.

12. Given $\sin A = \frac{2}{3}$; to find the other functions.

$$\textit{Ans.}\ \cos A = \frac{\sqrt{5}}{3}, \quad \tan A = \frac{2}{\sqrt{5}} = \tfrac{2}{5}\sqrt{5},$$

$$\sec A = \frac{3}{\sqrt{5}} = \tfrac{3}{5}\sqrt{5}, \quad \cot A = \tfrac{1}{2}\sqrt{5}, \quad \csc A = \tfrac{3}{2}$$

13. $\tan B = \frac{15}{8}$; find the other functions of the angle B.

Ans. $\sin B = \frac{15}{17}$, $\cos B = \frac{8}{17}$, $\cot B = \frac{8}{15}$, $\sec B = \frac{17}{8}$.

14. $A = \sin^{-1}\frac{1}{2}$; find the functions of A.

Ans. $\sin A = \frac{1}{2}$, $\cos A = \frac{1}{2}\sqrt{3}$, $\tan A = \frac{1}{3}\sqrt{3}$, $\csc A = 2$, etc.

15. Show that

$$\sin^{-1}\tfrac{5}{13} = \cos^{-1}\tfrac{12}{13} = \tan^{-1}\tfrac{5}{12}.$$

16. $\sin^{-1}\frac{8}{17} = \cos^{-1} x = \tan^{-1} y$; find x and y.

Ans. $x = \frac{15}{17}$, $y = \frac{8}{15}$.

17. If $y = \sin x$, show that $x = \cos^{-1}\sqrt{1 - y^2} = \tan^{-1}\dfrac{y}{\sqrt{1 - y^2}}$.

18. Show graphically that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = 45°$. (Suggestion. Construct $\tan^{-1}\frac{1}{2}$ and $\tan^{-1}\frac{1}{3}$, measure each and add the results.)

10. Functions of Complementary Angles. Let ABC be any right triangle, C the right angle, and a, b, c the three sides. Then angle A plus angle B equals 90, that is, A and B are complementary angles. Now

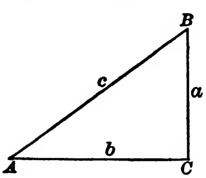


Fig. 13.

$$\sin A = \frac{a}{c}, \text{ and also } \cos B = \frac{a}{c},$$

therefore

$$\sin A = \frac{a}{c} = \cos B = \cos(90° - A).$$

Similarly

$$\cos A = \frac{b}{c} = \sin B = \sin\,(90^\circ - A),$$

$$\tan A = \frac{a}{b} = \cot B = \cot\,(90^\circ - A),$$

$$\cot A = \frac{b}{a} = \tan B = \tan\,(90^\circ - A),$$

$$\sec A = \frac{c}{b} = \csc B = \csc\,(90^\circ - A),$$

$$\csc A = \frac{c}{a} = \sec B = \sec\,(90^\circ - A).$$

These results may be put in words as follows:

The sine of an angle equals the cosine of the complement of the angle. The cosine of an angle equals the sine of the complement of the angle. The tangent of an angle equals the cotangent of the complement of the angle, and similarly for each of the remaining functions.

If we arrange the six functions in three pairs, viz., sine, cosine; tangent, cotangent; secant, cosecant; and call either function of a pair the cofunction of the other, we may express the six rules just given by a single rule, namely:

*Any cofunction of an angle is equal to the corresponding function of the complement of that angle.**

By means of this rule, any function of an acute angle can be expressed as some function of an angle less than 45°. Thus

$$\sin 75^\circ = \cos\,(90^\circ - 75^\circ) = \cos 15^\circ,$$
$$\cos 80^\circ = \sin\,(90^\circ - 80^\circ) = \sin 10^\circ, \text{etc.}$$

EXAMPLE. Given $\cot A = \tan 8A$, to find one value of A.

Solution. $\cot A = \tan\,(90^\circ - A)$, and from the condition of the problem

$$\cot A = \tan 8A,$$

therefore $$\tan\,(90^\circ - A) = \tan 8A.$$

* The term cosine was not used until the beginning of the 17th century. Before that time the term sine of the complement (*complimenti sinus*) was used instead a contraction of which gave rise to the present name cosine. Similarly cotangent is a contraction of *complimenti tangens* and cosecant of *complimenti secans*. The abbreviations sin, cos, tan, etc., did not come into general use until the middle of the 18th century.

This last equation is satisfied when

$$90^\circ - A = 8A.$$

Solving
$$A = 10^\circ.$$

NOTE. This is not the only value that A may have. After the definitions of the functions have been extended to angles greater than 90°, it will be seen that 30°, 50°, 70°, etc., are other values of A satisfying the equation $\cot A = \tan 8A$.

11. Functions of 0°, 30°, 45°, 60°, 90°. There are certain values of the angle for which the values of the functions may be easily determined exactly.

a. The functions of 0°. Let A be the angle formed by a fixed line OX and a line OP of constant length h rotating about O as a center. Let b be the base and a the altitude of the triangle formed by dropping a perpendicular from P to OX. As the angle increases, a increases and b decreases, and vice versa, as the angle decreases a decreases and b increases. As A approaches 0°, a approaches 0 and b approaches h, hence in the limit

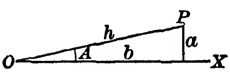


Fig. 14.

$$\sin 0^\circ = \frac{0}{h} = 0, \quad \cos 0^\circ = \frac{h}{h} = 1, \quad \tan 0^\circ = \frac{0}{h} = 0.$$

b. The functions of 30°. When the angle $A = 30^\circ$, the other acute angle of the right triangle equals 60°; the right triangle then forms one-half of an equilateral triangle. Each side of this triangle equals h, hence a, the altitude of the right triangle, equals $\frac{h}{2}$, and the base $= \sqrt{h^2 - \left(\frac{h}{2}\right)^2} = \sqrt{\frac{3}{4}h^2} = \frac{1}{2}h\sqrt{3}$, and we have

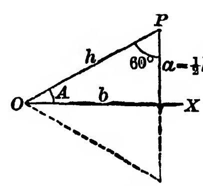


Fig. 15.

$$\sin 30^\circ = \frac{\frac{1}{2}h}{h} = \frac{1}{2}, \quad \cos 30^\circ = \frac{\frac{1}{2}h\sqrt{3}}{h} = \frac{1}{2}\sqrt{3},$$

$$\tan 30^\circ = \frac{\frac{1}{2}h}{\frac{1}{2}h\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}.$$

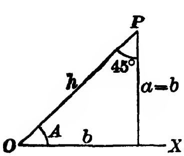


Fig. 16.

c. The functions of 45°. When $A = 45^\circ$ the right triangle is isosceles, that is, a and b are equal, and we have

$$h^2 = a^2 + a^2 = 2a^2, \quad a = h\sqrt{\tfrac{1}{2}} = \tfrac{1}{2}h\sqrt{2}, \text{ so that}$$

15. It is shown in geometry that if the radius of a circle is divided into extreme and mean ratio, the greater segment will be equal to the chord subtending an arc of 36°; that is, if in Fig. 18 AO is divided at B so that $AO : OB = OB : BA$, then the chord AC, taken equal to OB, subtends an angle COA equal to 36°.

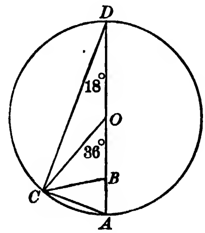


Fig. 18.

Put $AC = OB = x$, and $AO = r$, then CDA is a right triangle whose short side is x, whose hypotenuse is $2r$, and the angle $CDA = 18°$. Hence $\sin 18° = \frac{x}{2r}$, where x is the positive root of the equation

$$r : x = x : r - x.$$

Solving, we find the positive value of x, $x = r\left(-\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)$.

Hence $$\sin 18° = \frac{x}{2r} = \frac{r\left(-\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)}{2r} = \frac{1}{4}\left(-1 + \sqrt{5}\right).$$

Show now that $$\cos 18° = \frac{1}{4}\sqrt{10 + 2\sqrt{5}},\quad \sec 18° = \frac{\sqrt{10 - 2\sqrt{5}}}{\sqrt{5}},$$

$$\tan 18° = \frac{\sqrt{25 - 10\sqrt{5}}}{5},\quad \cot 18° = \sqrt{5 + 2\sqrt{5}}.$$

12. Fundamental Relations. In any right triangle (Fig. 19)

$$a^2 + b^2 = h^2.$$

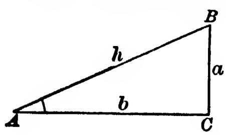


Fig. 19.

By dividing this equation first by h^2, then by b^2, and finally by a^2, we obtain in turn

$$\left(\frac{a}{h}\right)^2 + \left(\frac{b}{h}\right)^2 = 1, \quad \text{or} \quad \sin^2 A + \cos^2 A = 1, \qquad (1)$$

$$\left(\frac{a}{b}\right)^2 + 1 = \left(\frac{h}{b}\right)^2, \quad \text{or} \quad \tan^2 A + 1 = \sec^2 A, \qquad (2)$$

$$1 + \left(\frac{b}{a}\right)^2 = \left(\frac{h}{a}\right)^2, \quad \text{or} \quad 1 + \cot^2 A = \csc^2 A. \qquad (3)$$

(1), (2), and (3) are called the *square relations* of the trigonometric functions. These together with the three *reciprocal relations*

$$\sin A \cdot \csc A = 1, \qquad (4)$$

$$\cos A \cdot \sec A = 1, \qquad (5)$$

$$\tan A \cdot \cot A = 1, \qquad (6)$$

constitute the six fundamental relations of the trigonometric functions. To these, two others are usually added, viz.,

$$\tan A = \frac{\sin A}{\cos A}, \qquad \cot A = \frac{\cos A}{\sin A}. \tag{7}$$

Proof.

$$\frac{\sin A}{\cos A} = \frac{\frac{a}{h}}{\frac{b}{h}} = \frac{a}{b} = \tan A, \text{ and } \frac{\cos A}{\sin A} = \frac{1}{\tan A} = \cot A.$$

13. To express each of the Functions in Terms of a Given One.

a. Analytic method. EXAMPLE 1. To express each of the functions in terms of the sine.

$$\cos A = \sqrt{1 - \sin^2 A} \qquad \text{(By (1), Art. 12)}$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sqrt{1 - \sin^2 A}} \qquad \text{(By (7), Art. 12)}$$

$$\sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{1 - \sin^2 A}} \qquad \text{(By (5), Art. 12)}$$

$$\csc A = \frac{1}{\sin A} \qquad \text{(By (4), Art. 12)}$$

$$\cot A = \frac{1}{\tan A} = \frac{\sqrt{1 - \sin^2 A}}{\sin A} \qquad \text{(By (7), Art. 12)}$$

EXAMPLE 2. To express each of the functions in terms of the tangent.

$$\sec A = \sqrt{1 + \tan^2 A} \qquad \text{(By (2), Art. 12)}$$

$$\cot A = \frac{1}{\tan A} \qquad \text{(By (6), Art. 12)}$$

$$\cos A = \frac{1}{\sec A} = \frac{1}{\sqrt{1 + \tan^2 A}} \qquad \text{(By (5), Art. 12)}$$

$$\sin A = \cos A \cdot \tan A = \frac{\tan A}{\sqrt{1 + \tan^2 A}} \qquad \text{(By (7), Art. 12)}$$

$$\csc A = \frac{1}{\sin A} = \frac{\sqrt{1 + \tan^2 A}}{\tan A} \qquad \text{(By (4), Art. 12)}$$

b. Geometric method. EXAMPLE 1. To express each of the functions in terms of the sine.

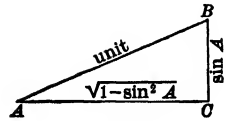


Fig. 20.

In the right triangle ABC (Fig. 20),

$$\sin A = \frac{BC}{AB}; \text{ hence if } AB \text{ is chosen for the unit of measure,}$$

$$\sin A = \frac{BC}{1} = BC, \text{ and } AC = \sqrt{AB^2 - BC^2} = \sqrt{1 - \sin^2 A}.$$

Then from the definitions of the trigonometric functions

$$\cos A = \frac{AC}{AB} = \sqrt{1 - \sin^2 A}, \quad \tan A = \frac{BC}{AC} = \frac{\sin A}{\sqrt{1 - \sin^2 A}}, \text{ etc.}$$

EXAMPLE 2. To express each of the functions in terms of the tangent.

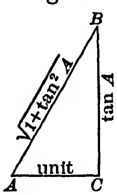


Fig. 21.

In Fig. 21,

$$\tan A = \frac{BC}{AC}; \text{ hence if } AC \text{ is chosen for the unit of measure,}$$

$$\tan A = \frac{BC}{1} = BC, \text{ and } AB = \sqrt{AC^2 + BC^2} = \sqrt{1 + \tan^2 A}.$$

Then by definition

$$\sin A = \frac{BC}{AB} = \frac{\tan A}{\sqrt{1 + \tan^2 A}}, \quad \cos A = \frac{AC}{AB} = \frac{1}{\sqrt{1 + \tan^2 A}}, \text{ etc.}$$

EXERCISE 8

1. From (1) show that

$$\sin A = \sqrt{1 - \cos^2 A}, \quad \cos A = \sqrt{1 - \sin^2 A},$$
$$(1 - \sin A)(1 + \sin A) = \cos^2 A,$$
$$(1 - \cos A)(1 + \cos A) = \sin^2 A, \quad \frac{\cos A}{1 - \sin A} = \frac{1 + \sin A}{\cos A},$$
$$\frac{\sin A}{1 - \cos A} = \frac{1 + \cos A}{\sin A}.$$

2. From (2) show that

$$\sec A = \sqrt{1 + \tan^2 A}, \quad \tan A = \sqrt{\sec^2 A - 1},$$
$$(\sec A - \tan A)(\sec A + \tan A) = 1.$$

3. Show that

$$\csc A = \sqrt{1 + \cot^2 A}, \quad \cot A = \sqrt{\csc^2 A - 1},$$
$$(\csc A - \cot A)(\csc A + \cot A) = 1.$$

4. From (7) show that

$$\cos A \cdot \tan A = \sin A, \quad \sin A \cdot \cot A = \cos A,$$
$$\sin A \cdot \sec A = \tan A, \quad \cos A \cdot \csc A = \cot A.$$

5. Express in words the relations given by formulas (1) to (7).

6. Use the analytic method to express the tangent in terms of the cosecant; in terms of the cosine.

7. Use the geometric method to express each the sine and the cosine in terms of the tangent.

In the following exercises compare your results with those given in the table on page 35.

8. Express each of the functions in terms of the cosine.

9. Express each of the functions in terms of the cosecant.

10. Express each of the functions in terms of the secant.

11. Use $\sin 30° = \frac{1}{2}$ to find each of the remaining functions of 30°.

12. $\sin 15° = \frac{1}{2}\sqrt{2 - \sqrt{3}}$; find the other functions of 15°.

Ans. $\cos 15° = \frac{1}{2}\sqrt{2+\sqrt{3}}$, $\tan 15° = 2 - \sqrt{3}$, $\sec 15° = 2\sqrt{2-\sqrt{3}}$,
$\csc 15° = 2\sqrt{2+\sqrt{3}}$, $\cot 15° = 2 + \sqrt{3}$.

13. If

$$\sin x = \sqrt{\frac{(s-b)(s-c)}{bc}}, \text{ where } s = \frac{a+b+c}{2},$$

prove that $\cos x = \sqrt{\frac{s(s-a)}{bc}}$ and $\tan x = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$.

14. Reduction of Trigonometric Expressions to their Simplest Form. Like algebraic expressions, expressions involving trigonometric functions may frequently be reduced to a simpler form. As a rule the reduction is most easily effected by expressing each of the functions which occur in the expression in terms of the sine and cosine and by reducing the resulting expression like any algebraic expression, treating the sine and cosine as two separate quantities. In the end the result may again be expressed in terms of whatever function or functions give the result the simplest form. Of course one might express everything in terms of a single function, say the

sine, but that would introduce radicals for every function except the cosecant. By using both the sine and cosine radicals are avoided, unless, of course, the expression involves radicals to begin with.

EXAMPLE 1. Reduce the expression $\sin A + \cot A \cos A$.

Solution. Substituting for $\cot A$ its value in terms of the sine and cosine, we have

$$\sin A + \frac{\cos A}{\sin A}\cos A.$$

Reducing to a common denominator,

$$\frac{\sin A \sin A + \cos A \cos A}{\sin A} = \frac{\sin^2 A + \cos^2 A}{\sin A}.$$

Now by (1), Art. 12, $\sin^2 A + \cos^2 A = 1$; hence finally,

$$\sin A + \cot A \cos A = \frac{1}{\sin A} = \csc A.$$

EXAMPLE 2. Reduce the expression

$$\sin^2 A \tan A + \cos^2 A \cot A + 2 \sin A \cos A.$$

Solution. Substituting for $\tan A$ and $\cot A$ their values in terms of the sine and cosine, (7), Art. 12, we have

$$\sin^2 A \frac{\sin A}{\cos A} + \cos^2 A \frac{\cos A}{\sin A} + 2 \sin A \cos A$$

$$= \frac{\sin^4 A + \cos^4 A + 2 \sin^2 A \cos^2 A}{\sin A \cos A} = \frac{(\sin^2 A + \cos^2 A)^2}{\sin A \cos A}$$

$$= \frac{1}{\sin A \cos A} = \csc A \sec A.$$

EXAMPLE 3. Reduce the expression

$$(\cos x \cos y - \sin x \sin y)^2 + (\sin x \cos y + \cos x \sin y)^2.$$

Solution. Squaring the expressions enclosed in parentheses,

$$(\cos x \cos y - \sin x \sin y)^2 = \cos^2 x \cos^2 y - 2 \cos x \cos y \sin x \sin y + \sin^2 x \sin^2 y,$$

$$(\sin x \cos y + \cos x \sin y)^2 = \sin^2 x \cos^2 y + 2 \sin x \cos y \cos x \sin y + \cos^2 x \sin^2 y.$$

Adding the right-hand members of the last two expressions, we have

$$\cos^2 y (\cos^2 x + \sin^2 x) + \sin^2 y (\sin^2 x + \cos^2 x)$$
$$= \cos^2 y + \sin^2 y = 1.$$

Complicated expressions, involving a single angle, may be most easily reduced by putting for each function its value in terms of the sides a, b, h of a right triangle. The resulting expression may then be reduced like any other algebraic expression. Of course, the relation $h^2 = a^2 + b^2$ may be made use of whenever an advantage is gained by it.

EXAMPLE 4. Reduce the expression

$$\sin^2 A \tan A + \cos^2 A \cot A + 2 \sin A \cos A.$$

Solution. Putting

$$\sin A = \frac{a}{h}, \quad \cos A = \frac{b}{h}, \quad \tan A = \frac{a}{b}, \quad \cot A = \frac{b}{a},$$

we have

$$\frac{a^2}{h^2}\frac{a}{b} + \frac{b^2}{h^2}\frac{b}{a} + 2\frac{a}{h}\frac{b}{h} = \frac{a^4 + b^4 + 2a^2b^2}{abh^2} = \frac{(a^2+b^2)^2}{abh^2} = \frac{a^2+b^2}{ab} = \frac{a}{b} + \frac{b}{a}$$
$$= \tan A + \cot A.$$

The preceding methods are perfectly general, but frequently it is of advantage to use other expedients. Any one of the seven relations in Art. 12 may be employed in the reduction. Sometimes the denominator may be simplified or removed by multiplying both terms of the fraction by a binominal factor like $1 - \cos A$, $1 + \sin A$, $\sec A - \tan A$, $\csc A + \cot A$; in short, as in algebra, the form of the expression may be changed by any operation which does not change the value of the result. Radical expressions should be avoided whenever possible.

EXAMPLE 5. Reduce the expression

$$\frac{\sin^2 x}{1 - \cos x}.$$

Solution. $$\frac{\sin^2 x}{1 - \cos x} \cdot \frac{1 + \cos x}{1 + \cos x} = \frac{\sin^2 x\,(1 + \cos x)}{1 - \cos^2 x}$$
$$= 1 + \cos x, \text{ since } 1 - \cos^2 x = \sin^2 x.$$

EXAMPLE 6. Reduce $\dfrac{\cos\theta}{\sec\theta + \tan\theta}$.*

* Greek letters are frequently used to represent angles. For the benefit of those students who do not already know the Greek letters and their names, the Greek alphabet has been printed in the front part of this book. The Greek letters are written as they are printed.

Solution. Multiplying both terms of the fraction by $\sec\theta - \tan\theta$, we have

$$\frac{\cos\theta}{\sec\theta+\tan\theta} = \frac{\cos\theta}{(\sec\theta+\tan\theta)}\cdot\frac{(\sec\theta-\tan\theta)}{(\sec\theta-\tan\theta)} = \frac{\cos\theta\,(\sec\theta-\tan\theta)}{\sec^2\theta-\tan^2\theta}$$
$$= \cos\theta\,(\sec\theta - \tan\theta),\quad \text{since } \sec^2\theta - \tan^2\theta = 1$$
$$= 1 - \sin\theta,\quad \text{since } \cos\theta\cdot\sec\theta = 1,\ \cos\theta\cdot\tan\theta = \sin\theta.$$

Exercise 9

Reduce the following expressions,—

1. $\dfrac{\cos A}{\sin A}\cdot\dfrac{1}{\cot^2 A}$. *Ans.* $\tan A$.
2. $\dfrac{\sin A}{\csc A}+\dfrac{\cos A}{\sec A}$. *Ans.* 1.
3. $\tan^2 x\csc^2 x - 1$. *Ans.* $\tan^2 x$.
4. $\dfrac{\cos y}{1-\tan y}-\dfrac{\sin y}{\cot y - 1}$. *Ans.* $\sin y + \cos y$.
5. $(\tan\theta + \cot\theta)\sin\theta\cos\theta$. *Ans.* 1.
6. $\cos x\tan x + \sin x\cot x$. *Ans.* $\sin x + \cos x$.
7. $\sec\alpha - \tan\alpha\sin\alpha$. *Ans.* $\cos\alpha$.
8. $\cot\theta + \dfrac{\sin\theta}{1+\cos\theta}$. *Ans.* $\csc\theta$.
9. $\sin^4 B + \cos^4 B + 2\cos^2 B\sin^2 B$. *Ans.* 1
10. $\dfrac{\cos^2 x}{1-\sin x}$. *Ans.* $1 + \sin x$.
11. $\sin A\,(\sec A + \csc A) - \cos A\,(\sec A - \csc A)$. *Ans.* $\sec A\csc A$.
12. $\cot\beta - \sec\beta\csc\beta\,(1 - 2\sin^2\beta)$. *Ans.* $\tan\beta$.
13. $(1+\sin A)(\sec A - \tan A)$. *Ans.* $\cos A$.
14. $\dfrac{\tan A + \tan B}{\cot A + \cot B}$. *Ans.* $\tan A\tan B$.
15. $(\sin x + \cos x)^2 + (\sin x - \cos x)^2$. *Ans.* 2.
16. $2\,(\sin^6\theta + \cos^6\theta) - 3\,(\sin^4\theta + \cos^4\theta) + 1$. *Ans.* 0.
17. $2\operatorname{vers} A - \operatorname{vers}^2 A$. *Ans.* $\sin^2 A$.

18. $\dfrac{\cos A \cot A - \sin A \tan A}{\csc A - \sec A}$. *Ans.* $1 + \sin A \cos A$.

19. $\dfrac{\sin B}{1 + \cos B} + \dfrac{1 + \cos B}{\sin B}$. *Ans.* $2 \csc B$.

20. $\csc^4 x\,(1 - \cos^4 x) - 2 \cot^2 x$. *Ans.* 1.

21. $(\cos x \cos y + \sin x \sin y)^2 + (\sin x \cos y - \cos x \sin y)^2$. *Ans.* 1.

22. $(x \cos \alpha - y \sin \alpha)^2 + (x \sin \alpha + y \cos \alpha)^2$. *Ans.* $x^2 + y^2$.

23. $\dfrac{\sec^2 A \sin^2 A - \csc^2 A + \csc^2 A \cos^2 A}{\sec^2 A \sin^2 A - \csc^2 A \cos^2 A}$. *Ans.* $\sin^2 A$.

24. $\sqrt{\dfrac{1 - \sin A}{1 + \sin A}}$. *Ans.* $\sec A - \tan A$.

15. Trigonometric Identities. Equations which express general relations, that is, equations which remain true no matter what values be given to the quantities which are considered variable, are called *identities*. $(x + 1)^2 = x^2 + 2x + 1$ is an identity, because it is true no matter what value be given to x. $x^2 - 5x + 6 = 0$ is not an identity, since it is not true unless x has the value 2 or 3. Similarly, $\sin^2 x + \cos^2 x = 1$ and $\tan A = \dfrac{\sin A}{\cos A}$ are identities, for they express general relations, relations which hold true no matter what x or A may be. On the other hand, $\sin^2 x - \frac{1}{2} \sin x = -\frac{1}{4}$ is not an identity, for it is true only when $\sin x = \frac{1}{2}$, that is, when $x = \sin^{-1} \frac{1}{2}$. To distinguish equations which are not identities from those which are, the former are sometimes called equations of condition, since they express conditions to which the variables are restricted and not general relations as do identities.

The fundamental trigonometric identities are given in Art. 12. All other trigonometric identities may be derived from these by properly combining them. To prove a given identity it is sufficient to reduce both sides to the same form. If no shorter way suggests itself, this may always be accomplished by reducing each side to its simplest form.

EXAMPLE 1. Prove the identity

$$(1 - \tan A)(1 - \cot A) + \sec A \csc A = 2.$$

Solution. Putting

$$\tan A = \frac{\sin A}{\cos A}, \quad \cot A = \frac{\cos A}{\sin A}, \quad \sec A = \frac{1}{\cos A}, \quad \csc A = \frac{1}{\sin A},$$

the left-hand member becomes

$$\left(1-\frac{\sin A}{\cos A}\right)\left(1-\frac{\cos A}{\sin A}\right)+\frac{1}{\cos A}\cdot\frac{1}{\sin A}$$

$$=\frac{(\cos A-\sin A)(\sin A-\cos A)+1}{\cos A\sin A}$$

$$=\frac{2\sin A\cos A-\sin^2 A-\cos^2 A+1}{\cos A\sin A}.$$

$$=\frac{2\sin A\cos A}{\cos A\sin A}=2.$$

If we had put $\tan A=\frac{a}{b}$, $\cot A=\frac{b}{a}$, $\sec A=\frac{h}{b}$, $\csc A=\frac{h}{a}$, the reduction would have been as follows,

$$\left(1-\frac{a}{b}\right)\left(1-\frac{b}{a}\right)+\frac{h}{b}\frac{h}{a}=\frac{(b-a)(a-b)+h^2}{ab}$$

$$=\frac{2ab-a^2-b^2+h^2}{ab}=\frac{2ab}{ab}=2.$$

EXAMPLE 2. Show that

$$\sin x\tan^2 x+\csc x\sec^2 x=2\tan x\sec x+\csc x-\sin x.$$

Solution. Reducing each member of the equation separately, we have

$$\sin x\tan^2 x+\csc x\sec^2 x=\sin x\frac{\sin^2 x}{\cos^2 x}+\frac{1}{\sin x}\frac{1}{\cos^2 x}=\frac{\sin^4 x+1}{\sin x\cos^2 x}.$$

Likewise the second member becomes

$$2\tan x\sec x+\csc x-\sin x=\frac{2\sin x}{\cos x}\cdot\frac{1}{\cos x}+\frac{1}{\sin x}-\sin x$$

$$=\frac{2\sin^2 x+\cos^2 x-\sin^2 x\cos^2 x}{\sin x\cos^2 x}$$

$$=\frac{2\sin^2 x+(1-\sin^2 x)\cos^2 x}{\sin x\cos^2 x}$$

$$=\frac{2\sin^2 x+(1-\sin^2 x)(1-\sin^2 x)}{\sin x\cos^2 x}$$

$$=\frac{\sin^4 x+1}{\sin x\cos^2 x}.$$

EXAMPLE 3. Is $\sec^2 A \csc^2 A = \tan^2 A + \cot^2 A + 2$?

Solution.

$$\begin{aligned}\sec^2 A \csc^2 A &= (1 + \tan^2 A)(1 + \cot^2 A)\\ &= 1 + \tan^2 A + \cot^2 A + \tan^2 A \cot^2 A\\ &= 1 + \tan^2 A + \cot^2 A + 1, \text{ since } \tan A \cot A = 1.\\ &= \tan^2 A + \cot^2 A + 2.\end{aligned}$$

Hence the identity is true.

EXERCISE 10

Prove the following identities:

1. $\sin^2 A \sec^2 A = \sec^2 A - 1$.
2. $(\sin B + \cos B)^2 = 1 + 2 \sin B \cos B$.
3. $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x = 1 - 2\cos^2 x = 2\sin^2 x - 1$.
4. $(\sin^2 \theta - \cos^2 \theta)^2 = 1 - 4 \sin^2 \theta \cos^2 \theta$.
5. $\dfrac{\cot A \cos A}{\cot A + \cos A} = \dfrac{\cot A - \cos A}{\cot A \cos A}$.
6. $\sin^3 x + \cos^3 x = (\sin x + \cos x)(1 - \sin x \cos x)$.
7. $(\sin A + \cos A)(\tan A + \cot A) = \sec A + \csc A$.
8. $\dfrac{\sec\theta + \csc\theta}{\sec\theta - \csc\theta} = \dfrac{\tan\theta + 1}{\tan\theta - 1} = \dfrac{1 + \cot\theta}{1 - \cot\theta}$.
9. $\sec^4 A + \tan^4 A = 1 + 2 \sec^2 A \tan^2 A$.
10. $\tan\alpha + \tan\beta = \tan\alpha \tan\beta (\cot\alpha + \cot\beta)$.
11. $(1 + \sin x + \cos x)^2 = 2(1 + \sin x)(1 + \cos x)$.
12. $\dfrac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \tan A + \sec A$.
13. $\cos^4 x - \sin^4 x = \cos^2 x (1 - \tan x)(1 + \tan x)$.
14. $\cos^2 x + \sin^2 x \sin^2 y + \sin^2 x \cos^2 y = 1$.
15. $(\cos\alpha \cos\beta + \sin\alpha \sin\beta \cos\gamma)^2 + (\sin\alpha \cos\beta - \cos\alpha \sin\beta \cos\gamma)^2$ $= 1 - \sin^2\beta \sin^2\gamma$.

	$\sin A$	$\cos A$	$\tan A$	$\csc A$	$\sec A$	$\cot A$
$\sin A =$		$\sqrt{1-\cos^2 A}$	$\dfrac{\tan A}{\sqrt{1+\tan^2 A}}$	$\dfrac{1}{\csc A}$	$\dfrac{\sqrt{\sec^2 A-1}}{\sec A}$	$\dfrac{1}{\sqrt{1+\cot^2 A}}$
$\cos A =$	$\sqrt{1-\sin^2 A}$		$\dfrac{1}{\sqrt{1+\tan^2 A}}$	$\dfrac{\sqrt{\csc^2 A-1}}{\csc A}$	$\dfrac{1}{\sec A}$	$\dfrac{\cot A}{\sqrt{1+\cot^2 A}}$
$\tan A =$	$\dfrac{\sin A}{\sqrt{1-\sin^2 A}}$	$\dfrac{\sqrt{1-\cos^2 A}}{\cos A}$		$\dfrac{1}{\sqrt{\csc A^2-1}}$	$\sqrt{\sec^2 A-1}$	$\dfrac{1}{\cot A}$
$\csc A =$	$\dfrac{1}{\sin A}$	$\dfrac{1}{\sqrt{1-\cos^2 A}}$	$\dfrac{\sqrt{1+\tan^2 A}}{\tan A}$		$\dfrac{\sec A}{\sqrt{\sec^2 A-1}}$	$\sqrt{1+\cot^2 A}$
$\sec A =$	$\dfrac{1}{\sqrt{1-\sin^2 A}}$	$\dfrac{1}{\cos A}$	$\sqrt{1+\tan^2 A}$	$\dfrac{\csc A}{\sqrt{\csc^2 A-1}}$		$\dfrac{\sqrt{1+\cot^2 A}}{\cot A}$
$\cot A =$	$\dfrac{\sqrt{1-\sin^2 A}}{\sin A}$	$\dfrac{\cos A}{\sqrt{1-\cos^2 A}}$	$\dfrac{1}{\tan A}$	$\sqrt{\csc^2 A-1}$	$\dfrac{1}{\sqrt{\sec^2 A-1}}$	

CHAPTER III

SOLUTION OF RIGHT TRIANGLES BY NATURAL FUNCTIONS

16. Tables of Natural Functions. In the preceding chapter we computed the functions of 30°, 45° and 60°. Later we shall learn how to compute the functions of any given angle. Now we cannot remember the values of the functions for all angles, and the computation by which the values are determined is too laborious to be repeated each time the value of a particular function is needed. For this reason the values of the functions for every degree, minute and second from 0° to 90° have been computed once for all and the results tabulated in tables, known as *tables of natural functions.* Usually such tables contain the sines, cosines, tangents and cotangents of angles differing by 1 minute; in some tables the angles differ by 10 seconds, and in still others by only 1 second. From such tables, the value of the sine, cosine, tangent and cotangent may be found whenever needed, and conversely, when the value of a function is known, the corresponding angle may be found by use of the tables. Secants and cosecants are not given directly by the tables, but may be found indirectly from the cosines and sines respectively, since $\sec A = \frac{1}{\cos A}$, and $\csc A = \frac{1}{\sin A}$.

17. To Find the Natural Functions of an Angle Less than 90°.
a. When the angle is less than 45°, the degrees are found at the head of the column and the minutes in the left-hand column. The number, in the same horizontal line as the minutes and in the same column as the degrees, is that function of the angle whose name appears at the head of the column.

EXAMPLE 1. To find the sine of 26° 31′.

Solution. We find the column in the table of natural sines and cosines (specimen page, p. 40) which is headed by 26° and follow down the column marked sine *on top* until we come to the number 4465 which is in the line beginning with 31′. The number 4465 considered a decimal is the required sine, that is,

$$\sin 26^\circ 31' = 0.4465.$$

b. When the angle is greater than 45°, the degrees are found at the foot of the column and the minutes in the right-hand column. The number in the same horizontal line as the minutes in the right-hand column and in the same column as the degrees at the bottom, is that function of the angle whose name appears at the foot of the column.

EXAMPLE 2. To find cos 63° 29′.

Solution. We find that column in the table of natural sines and cosines (see specimen page) which has 63° at its foot, and *follow up* the column marked cos *at the bottom* until we come to the number 4465 which is in the horizontal line ending with 29′. The number 4465 considered a decimal is the required cosine, that is,

$$\cos 63° \, 29' = 0.4465.$$

c. When the angle consists of degrees, minutes and seconds, the result obtained for the given degrees and minutes must be corrected for the additional seconds.

EXAMPLE 3. To find sin 28° 46′ 36″.

Solution. $28° \, 46' \, 36'' = 28° \, 46.6'$.

From the table we find

$$\begin{array}{r} \sin 28° \, 46' = 0.4812 \\ \sin 28° \, 47' = 0.4815 \\ \hline \text{difference for } 1' = 0.0003 \end{array}$$

The angle 28° 46.6′ whose sine we seek is $\frac{6}{10}$ the way between the two angles 28° 46′ and 28° 47′, hence we take for its sine the sine of 28° 46′ increased by $\frac{6}{10}$ of the difference between sin 28° 46′ and sin 28° 47′. $\frac{6}{10}$ of $0.0003 = 0.00018$, or 0.0002 if we carry 4 decimal places only. Hence

$$\sin 28° \, 46' \, 36'' = 0.4812 + 0.0002 = 0.4814.$$

EXAMPLE 4. To find cos 61° 13′ 24″.

Solution. $61° \, 13' \, 24'' = 61° \, 13.4'$.

From the table we find

$$\begin{array}{r} \cos 61° \, 13' = 0.4815 \\ \cos 61° \, 14' = 0.4812 \\ \hline \text{difference for } 1' = 0.0003 \end{array}$$

$\frac{4}{10}$ of $0.0003 = 0.00012$, or 0.0001 if we carry four places only, and since the cosine of 61° 13.4′ must be between cos 61° 13′ and cos 61° 14′,

$$\cos 61° \, 13' \, 24'' = 0.4815 - 0.0001 = 0.4814.$$

Observe that in Example 3 the difference was *added to the sine* of the smaller angle, while in Example 4 the difference was *subtracted from the cosine* of the smaller angle. This is because as the angle increases the *sine increases* while the *cosine decreases.* For the same reason the difference must be *added to the tangent* and *subtracted from the cotangent* of the smaller angle.

The *tangent or cotangent of an angle* is found from the table of natural tangents and cotangents in exactly the same way that the sine or cosine is found from the table of natural sines and cosines.

EXAMPLE 5. To find $\tan 63° 16' 32''$.

Solution. $63° 16' 32'' = 63° 16.5\frac{1}{3}'$.

From the table of natural tangents and cotangents (see specimen page, p. 41) we find

$$\begin{array}{r} \tan 63° 16' = 1.9854 \\ \underline{\tan 63° 17' = 1.9868} \\ \text{difference for } 1' = 0.0014 \end{array}$$

$5\frac{1}{3}$ times $0.0014 = 0.000747$, or 0.0007 to four places. Hence

$$\tan 63° 16' 32'' = 1.9854 + 0.0007 = 1.9861.$$

EXAMPLE 6. To find $\cot 26° 43' 28''$.

Solution. $26° 43' 28'' = 26° 43.4\frac{2}{3}'$.

From the table (specimen page)

$$\begin{array}{r} \cot 26° 43' = 1.9868 \\ \underline{\cot 26° 44' = 1.9854} \\ \text{difference for } 1' = 0.0014 \end{array}$$

$4\frac{2}{3}$ of $0.0014 = 0.00065$ or 0.0007 to four places. Hence

$$\cot 26° 43' 28'' = 1.9868 - 0.0007 = 1.9861.$$

The process of finding the value of a function of an angle intermediate to two consecutive angles whose functions are given directly in the table, is called *interpolation.* Thus in Example 4, $\cot 36° 43'$ and $\cot 36° 44'$ are given directly in the table, while $\cot 36° 43' 28''$, which is not given in the table, was found by interpolation.

In interpolating we assumed that the increase or decrease of the function is proportional to the increase of the angle, that is, we assumed what is known as the *principle of proportional parts.* Briefly stated it is this,—

*For small changes in the angle the change in the function of an angle is nearly * proportional to the change in the angle.*

EXAMPLE 7. One angle of a right triangle is $25° 48.5'$ and the hypotenuse is 235.0. Find the remaining sides of the triangle.

Solution. In the adjacent figure, let A represent the given angle and c the hypotenuse.

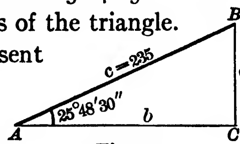


Fig. 22

To find a we have,

$$\frac{a}{c} = \sin A, \text{ or } a = c \sin A.$$

$$c = 235, \text{ and from the table we have } \sin A = 0.4354,$$

hence $$a = 235 \times 0.4354 = 102.3.$$

To find b, we have,

$$\frac{b}{c} = \cos A, \text{ or } b = c \cos A,$$

$$c = 235, \text{ and from the table we have } \cos A = 0.9003,$$

hence, $$b = 235 \times 0.9003 = 211.6.$$

To check our results, we use the relation $\frac{a}{b} = \tan A$, that is, if our results are correct, the quotient of a by b must agree with the value of $\tan A$ as found from the table.

$\frac{a}{b} = \frac{102.3}{211.6} = 0.4835$, while $\tan A$ as given in the table is 0.4836.

The difference of 1 in the last decimal place arises from the neglected parts of the decimals. If we had carried out the work to five significant figures instead of four, the quotient of a divided by b would have been 0.4836.

EXERCISE 11

All the functions called for in this exercise are found on the specimen pages of natural functions on pp. 39, 40.

1. Find $\sin 26°$, $\cos 27° 30'$, $\tan 25° 45'$, $\cot 29° 59'$.

   Ans. 0.4384, 0.8870, 0.4823, 1.7332.

2. Find $\cos 64°$, $\sin 62° 30'$, $\cot 64° 15'$, $\tan 60° 01'$.

   Ans. 0.4384, 0.8870, 0.4823, 1.7332.

* We say nearly, for no exact proportion exists, no matter how small the change in the angle. All we can say is, that when the change in the angle is small, the principle of proportional parts gives results which in most cases are sufficiently exact for practical purposes.

SPECIMEN PAGE

′	25° sin	25° cos	26° sin	26° cos	27° sin	27° cos	28° sin	28° cos	29° sin	29° cos	′
0	4226	9063	4384	8988	4540	8910	4695	8829	4848	8746	**60**
1	4229	9062	4386	8987	4542	8909	4697	8828	4851	8745	59
2	4231	9061	4389	8985	4545	8907	4700	8827	4853	8743	58
3	4234	9059	4392	8984	4548	8906	4702	8825	4856	8742	57
4	4237	9058	4394	8983	4550	8905	4705	8824	4858	8741	56
5	4239	9057	4397	8982	4553	8903	4708	8823	4861	8739	**55**
6	4242	9056	4399	8980	4555	8902	4710	8821	4863	8738	54
7	4245	9054	4402	8979	4558	8001	4713	8820	4866	8736	53
8	4247	9053	4405	8978	4561	8899	4715	8819	4868	8735	52
9	4250	9052	4407	8976	4563	8898	4718	8817	4871	8733	51
10	4253	9051	4410	8975	4566	8897	4720	8816	4874	8732	**50**
11	4255	9050	4412	8974	4568	8895	4723	8814	4876	8731	49
12	4258	9048	4415	8973	4571	8894	4726	8813	4879	8729	48
13	4260	9047	4418	8971	4574	8893	4728	8812	4881	8728	47
14	4263	9046	4420	8970	4576	8892	4731	8810	4884	8726	46
15	4266	9045	4423	8969	4579	8890	4733	8809	4886	8725	**45**
16	4268	9043	4425	8967	4581	8889	4736	8808	4889	8724	44
17	4271	9042	4428	8966	4584	8888	4738	8806	4891	8722	43
18	4274	9041	4431	8965	4586	8886	4741	8805	4894	8721	42
19	4276	9040	4433	8964	4589	8885	4743	8803	4896	8719	41
20	4279	9038	4436	8962	4592	8884	4746	8802	4899	8718	**40**
21	4281	9037	4439	8961	4594	8882	4749	8801	4901	8716	39
22	4284	9036	4441	8960	4597	8881	4751	8799	4904	8715	38
23	4287	9035	4414	8958	4599	8879	4754	8798	4907	8714	37
24	4289	9033	4446	8957	4602	8878	4756	8796	4909	8712	36
25	4292	9032	4449	8956	4605	8877	4759	8795	4912	8711	**35**
26	4295	9031	4452	8955	4607	8875	4761	8794	4914	8709	34
27	4297	9030	4454	8953	4610	8874	4764	8792	4917	8708	33
28	4300	9028	4457	8952	4612	8873	4766	8791	4919	8706	32
29	4302	9027	4459	8951	4615	8871	4769	8790	4922	8705	31
30	4305	9026	4462	8949	4617	8870	4772	8788	4924	8704	**30**
31	4308	9025	4465	8948	4620	8869	4774	8787	4927	8702	29
32	4310	9023	4467	8947	4623	8867	4777	8785	4929	8701	28
33	4313	9022	4470	8945	4625	8866	4779	8784	4932	8699	27
34	4316	9021	4472	8944	4628	8865	4782	8783	4934	8698	26
35	4318	9020	4475	8943	4630	8863	4784	8781	4937	8696	**25**
36	4321	9018	4478	8942	4633	8862	4787	8780	4939	8695	24
37	4323	9017	4480	8940	4636	8861	4789	8778	4942	8694	23
38	4326	9016	4483	8939	4638	8859	4792	8777	4944	8692	22
39	4329	9015	4485	8938	4641	8858	4795	8776	4947	8691	21
40	4331	9013	4488	8936	4643	8857	4797	8774	4950	8689	**20**
41	4334	9012	4491	8935	4646	8855	4800	8773	4952	8688	19
42	4337	9011	4493	8934	4648	8854	4802	8771	4955	8686	18
43	4339	9010	4496	8932	4651	8853	4805	8770	4957	8685	17
44	4342	9008	4498	8931	4654	8851	4857	8769	4960	8683	16
45	4344	9007	4501	8930	4656	8850	4810	8767	4962	8682	**15**
46	4347	9006	4504	8928	4659	8849	4812	8766	4965	8681	14
47	4350	9004	4506	8927	4661	8847	4815	8764	4967	8679	13
48	4352	9003	4509	8926	4664	8846	4818	8763	4970	8678	12
49	4355	9002	4511	8925	4666	8844	4820	8762	4972	8676	11
50	4358	9001	4514	8923	4669	8843	4823	8760	4975	8675	**10**
51	4360	8999	4517	8922	4672	8842	4825	8759	4977	8673	9
52	4363	8998	4519	8921	4674	8840	4828	8757	4980	8672	8
53	4365	8997	4522	8919	4677	8839	4830	8756	4982	8670	7
54	4368	8996	4524	8918	4679	8838	4833	8755	4985	8669	6
55	4371	8994	4527	8917	4682	8836	4835	8753	4987	8668	**5**
56	4373	8993	4530	8915	4684	8835	4838	8752	4990	8666	4
57	4376	8992	4532	8914	4687	8834	4840	8750	4992	8665	3
58	4378	8990	4535	8913	4690	8832	4843	8749	4995	8663	2
59	4381	8989	4537	8911	4692	8831	4846	8748	4997	8662	1
60	4384	8988	4540	8910	4695	8829	4848	8746	5000	8660	**0**
′	cos 64°	sin 64°	cos 63°	sin 63°	cos 62°	sin 62°	cos 61°	sin 61°	cos 60°	sin 60°	′

SPECIMEN PAGE

′	25°		26°		27°		28°		29°		′
	tan	cot	tan	cot	tan	cot	tan	cot	tan	cot	
0	4663	2.1445	4877	2.0503	5095	1.9626	5317	1.8807	5543	1.8040	**60**
1	4667	2.1429	4881	2.0488	5099	1.9612	5321	1.8794	5547	1.8028	59
2	4670	2.1413	4885	2.0473	5103	1.9598	5325	1.8781	5551	1.8016	58
3	4674	2.1396	4888	2.0458	5106	1.9584	5328	1.8768	5555	1.8003	57
4	4677	2.1380	4892	2.0443	5110	1.9570	5332	1.8755	5558	1.7991	56
5	4681	2.1364	4895	2.0428	5114	1.9556	5336	1.8741	5562	1.7979	**55**
6	4684	2.1348	4899	2.0413	5117	1.9542	5340	1.8728	5566	1.7966	54
7	4688	2.1332	4903	2.0398	5121	1.9528	5343	1.8715	5570	1.7954	53
8	4691	2.1315	4906	2.0383	5125	1.9514	5347	1.8702	5574	1.7942	52
9	4695	2.1299	4910	2.0368	5128	1.9500	5351	1.8689	5577	1.7930	51
10	4699	2.1283	4913	2.0353	5132	1.9486	5354	1.8676	5581	1.7917	**50**
11	4702	2.1267	4917	2.0338	5136	1.9472	5358	1.8663	5585	1.7905	49
12	4706	2.1251	4921	2.0323	5139	1.9458	5362	1.8650	5589	1.7893	48
13	4709	2.1235	4924	2.0308	5143	1.9444	5366	1.8637	5593	1.7881	47
14	4713	2.1219	4928	2.0293	5147	1.9430	5369	1.8624	5596	1.7868	46
15	4716	2.1203	4931	2.0278	5150	1.9416	5373	1.8611	5600	1.7856	**45**
16	4720	2.1187	4935	2.0263	5154	1.9402	5377	1.8598	5604	1.7844	44
17	4723	2.1171	4939	2.0248	5158	1.9388	5381	1.8585	5608	1.7832	43
18	4727	2.1155	4942	2.0233	5161	1.9375	5384	1.8572	5612	1.7820	42
19	4731	2.1139	4946	2.0219	5165	1.9361	5388	1.8559	5616	1.7808	41
20	4734	2.1123	4950	2.0204	5169	1.9347	5392	1.8546	5619	1.7796	**40**
21	4738	2.1107	4953	2.0189	5172	1.9333	5396	1.8533	5623	1.7783	39
22	4741	2.1092	4957	2.0174	5176	1.9319	5399	1.8520	5627	1.7771	38
23	4745	2.1076	4960	2.0160	5180	1.9306	5403	1.8507	5631	1.7759	37
24	4748	2.1060	4964	2.0145	5184	1.9292	5407	1.8495	5635	1.7747	36
25	4752	2.1044	4968	2.0130	5187	1.9278	5411	1.8482	5639	1.7735	**35**
26	4755	2.1028	4971	2.0115	5191	1.9265	5415	1.8469	5642	1.7723	34
27	4759	2.1013	4975	2.0101	5195	1.9251	5418	1.8456	5646	1.7711	33
28	4763	2.0997	4979	2.0086	5198	1.9237	5422	1.8443	5650	1.7699	32
29	4766	2.0981	4982	2.0072	5202	1.9223	5426	1.8430	5654	1.7687	31
30	4770	2.0965	4986	2.0057	5206	1.9210	5430	1.8418	5658	1.7675	**30**
31	4773	2.0950	4989	2.0042	5209	1.9196	5433	1.8405	5662	1.7663	29
32	4777	2.0934	4993	2.0028	5213	1.9183	5437	1.8392	5665	1.7651	28
33	4780	2.0918	4997	2.0013	5217	1.9169	5441	1.8379	5669	1.7639	27
34	4784	2.0903	5000	1.9999	5220	1.9155	5445	1.8367	5673	1.7627	26
35	4788	2.0887	5004	1.9984	5224	1.9142	5448	1.8354	5677	1.7615	**25**
36	4791	2.0872	5008	1.9970	5228	1.9128	5452	1.8341	5681	1.7603	24
37	4795	2.0856	5011	1.9955	5232	1.9115	5456	1.8329	5685	1.7591	23
38	4798	2.0840	5015	1.9941	5235	1.9101	5460	1.8316	5688	1.7579	22
39	4802	2.0825	5019	1.9926	5239	1.9088	5464	1.8303	5692	1.7567	21
40	4806	2.0809	5022	1.9912	5243	1.9074	5467	1.8291	5696	1.7556	**20**
41	4809	2.0794	5026	1.9897	5246	1.9061	5471	1.8278	5700	1.7544	19
42	4813	2.0778	5029	1.9883	5250	1.9047	5475	1.8265	5704	1.7532	18
43	4816	2.0763	5033	1.9868	5254	1.9034	5479	1.8253	5708	1.7520	17
44	4820	2.0748	5037	1.9854	5258	1.9020	5482	1.8240	5712	1.7508	16
45	4823	2.0732	5040	1.9840	5261	1.9007	5486	1.8228	5715	1.7496	**15**
46	4827	2.0717	5044	1.9825	5265	1.8993	5490	1.8215	5719	1.7485	14
47	4831	2.0701	5048	1.9811	5269	1.8980	5494	1.8202	5723	1.7473	13
48	4834	2.0686	5051	1.9797	5272	1.8967	5498	1.8190	5727	1.7461	12
49	4838	2.0671	5055	1.9782	5276	1.8953	5501	1.8177	5731	1.7449	11
50	4841	2.0655	5059	1.9768	5280	1.8940	5505	1.8165	5735	1.7437	**10**
51	4845	2.0640	5062	1.9754	5284	1.8927	5509	1.8152	5739	1.7426	9
52	4849	2.0625	5066	1.9740	5287	1.8913	5513	1.8140	5743	1.7414	8
53	4852	2.0609	5070	1.9725	5291	1.8900	5517	1.8127	5746	1.7402	7
54	4856	2.0594	5073	1.9711	5295	1.8887	5520	1.8115	5750	1.7391	6
55	4859	2.0579	5077	1.9697	5298	1.8873	5524	1.8103	5754	1.7379	**5**
56	4863	2.0564	5081	1.9683	5302	1.8860	5528	1.8090	5758	1.7367	4
57	4867	2.0549	5084	1.9669	5306	1.8847	5532	1.8078	5762	1.7355	3
58	4870	2.0533	5088	1.9654	5310	1.8834	5535	1.8065	5766	1.7344	2
59	4874	2.0518	5092	1.9640	5313	1.8820	5539	1.8053	5770	1.7332	1
60	4877	2.0503	5095	1.9626	5317	1.8807	5543	1.8040	5774	1.7321	**0**
	cot	tan	cot	tan	cot	tan	cot	tan	cot	tan	
′	64°		63°		62°		61°		60°		′

3. Find sin $28° 30' 24''$, tan $61° 24' 36''$, cos $60° 30' 25''$.
Ans. 0.4773, 1.8349, 0.4923.

4. Find cos $25° 50' 10''$, cot $28° 25' 50''$, tan $27° 00' 30''$.
Ans. 0.9001, 1.8471, 0.5097.

5. Find sec $25° 35'$, csc $25° 28'$. *Ans.* 1.1087, 2.3256.

Solve the following right triangles:

6. Given an acute angle $A = 26° 15'$ and the hypotenuse $c = 35.0$; to find the remaining parts.
Ans. $B = 63° 45'$, $a = 15.5$, $b = 31.4$.

7. Given $A = 28° 50'$ and the side opposite, $a = 150$.
Ans. $B = 61° 10'$, $b = 272$, $c = 311$.

8. Given $A = 63° 40' 30''$ and the side adjacent. $b = 363$. Find a and c and check your results.

18. To Find the Angle Less than 90° Corresponding to a Given Natural Function.

a. When the function is one of the numbers given in the table, the angle is found by taking the number of degrees which stands at the head or foot of the column according as the name of the function appears at the head or the foot of the column in which the number is found, and the number of minutes at the left or right end of the line in which the number is found according as the degrees have been taken from the top or the bottom of the column.

EXAMPLE 1. Find the angle whose sine is 0.4633.

Solution. We find the number 4633 in the column which has 27° written at its head and 62° at its foot, and in the line which begins with 36′ and ends with 24′ (see specimen page, p. 39). Since the given number is a sine, and the name "sin" appears at the head of the column in which 4633 is found, we take the degrees from the top of the column and the minutes from the left of the line in which 4633 is found, that is,

$$\text{the angle whose sine is } 0.4633, \text{ or } \sin^{-1} 0.4633 = 27° 36'.$$

EXAMPLE 2. Find the angle whose cosine is 0.4633.

Solution. This time the name of the function appears at the bottom of the column in which 4633 is found, hence

$$\text{the angle whose cosine is } 0.4633, \text{ or } \cos^{-1} 0.4633 = 62° 24'.$$

b. When the function is not given in the table, we find the corresponding angle by reversing the process by means of which we find the function when the angle is given.

EXAMPLE 3. $\tan x = 0.5492$, to find x.

Solution. The number 5492 is not found in the table of natural tangents and cotangents, but two other numbers are found (see specimen page), namely, 5490 and 5494, one of which is a little smaller, the other a little larger than the given number.

$$0.5490 = \tan 28^\circ 46'$$
$$0.5494 = \tan 28^\circ 47'.$$

It is plain, therefore, that x, the angle whose tangent is 0.5492, is somewhere between $28^\circ 46'$ and $28^\circ 47'$. Applying the principle of proportional parts we have,

$$\frac{x - 28^\circ 46'}{28^\circ 47' - 28^\circ 46'} = \frac{\tan x - \tan 28^\circ 46'}{\tan 28^\circ 47' - \tan 28^\circ 46'}$$

that is,

$$\frac{x - 28^\circ 46'}{1' \text{ or } 60''} = \frac{0.5492 - 0.5490}{0.5494 - 0.5490} = \frac{0.0002}{0.0004} = \frac{2}{4} = \frac{1}{2},$$

and solving for x,

$$x = 28^\circ 46' + \tfrac{1}{2} \text{ of } 60'' = 28^\circ 46' 30''.$$

It is not necessary to go through all this work each time. All we need to remember is that the smaller of the two angles between which x lies must be increased by

$$\frac{d}{D} \text{ of } 60'',$$

where

D is the difference (without regard to the decimal point) between the functions of the two angles between which x lies,

d is the difference (without regard to the decimal point) between the function of the smaller angle and the given function.

EXAMPLE 4. $\tan x = 1.9887$, to find x.

Solution. From the table (specimen page)

$$1.9883 = \tan 63^\circ 18'$$
$$1.9897 = \tan 63^\circ 19'$$
$$D = 19897 - 19883 = 14,$$
$$d = 19887 - 19883 = 4,$$
$$x = 63^\circ 18' + \tfrac{4}{14} \text{ of } 60'' = 63^\circ 18' 17''.$$

EXAMPLE 5. $\cos x = 0.4767$, to find x.

Solution. From the table

$$0.4769 = \cos 61^\circ 31'$$
$$0.4766 = \cos 61^\circ 32'$$
$$D = 4769 - 4766 = 3,$$
$$d = 4769 - 4767 = 2,$$
$$x = 61^\circ 31' + \tfrac{2}{3} \text{ of } 60'' = 61^\circ 31' 40''.$$

19. Accuracy of Results. When an angle has been obtained from a four-place table (a table giving four places of decimals only), the number of seconds in the angle found cannot be relied upon with certainty. This is best shown by considering a special example, as Example 4 above.

Since the tangents are given to four places only, the actual value of $\tan 63^\circ 18'$ is not necessarily 1.9883, but may have any value between 1.98825 and 1.98835.

Similarly the actual value of $\tan 63^\circ 19'$ may be any number between 1.98965 and 1.98975.

Hence D, the difference between $\tan 63^\circ 19'$ and $\tan 63^\circ 18'$, is not necessarily 0.0014, but may be any number

$$\text{less than } 1.98975 - 1.98825 = 0.00150,$$
$$\text{and greater than } 1.98965 - 1.98835 = 0.00130.$$

Again, $\tan x$, four places only being given, may have any value between 1.98865 and 1.98875, so that d, the difference between $\tan x$ and $\tan 63^\circ 18'$, may be any number

$$\text{less than } 1.98875 - 1.98825 = 0.00050,$$
$$\text{and greater than } 1.98865 - 1.98835 = 0.00030.$$

Hence,

$$\frac{d}{D} \text{ of } 60'' \text{ must be some number less than } \frac{50}{130} \text{ of } 60'' = 23'',$$
$$\text{and greater than } \frac{30}{150} \text{ of } 60'' = 12''.$$

The conclusion is that the result $63^\circ 18' 17''$ previously given is uncertain by at most $6''$.

From the example just given, it appears that the amount by which the result obtained from a table is uncertain depends upon the difference D, which varies not only for different functions of the same angle, but also for the same function of different angles. No general rule can be laid down to cover the amount of uncertainty in all cases. If absolute certainty in the number of seconds is required, a seven-place table should be used, giving the values of the functions from second to second for small angles, and for intervals of $10''$ for larger angles.

When a four-place table is used, and no special consideration is given to the nature of the differences involved, the number of seconds of an angle obtained by interpolation cannot be relied upon.

The following rules will be of some aid to beginners:

1. *An angle less than $45°$ can be obtained more accurately from a sine than from a cosine, while an angle greater than $45°$ can be obtained more accurately from a cosine than from a sine.*

This is because the sines of angles less than $45°$ vary more than the cosines, and at the same time the principle of proportional parts is more nearly true for sines of small angles than for cosines, while the opposite is true for angles greater than $45°$.

2. *Very small angles can be obtained with greater accuracy by interpolation from tangents than from cotangents, while angles near $90°$ can be obtained with greater accuracy from cotangents than from tangents.*

The reason for this lies in the fact that the principle of proportional parts ceases to apply to cotangents of small angles and to tangents of angles near $90°$.

Exercise 12

Find the following angles:

1. $\sin^{-1} 0.4904$, $\cos^{-1} 0.4904$, $\tan^{-1} 1.8940$, $\cot^{-1} 1.8940$.
Ans. $29° 22'$, $60° 38'$, $62° 10'$, $27° 50'$.

2. $\sin^{-1} 0.4267$, $\cos^{-1} 0.4900$, $\tan^{-1} 2.1036$, $\cot^{-1} 0.5644$.
Ans. $25° 15' 30''$, $60° 39' 30''$, $64° 34' 30''$, $60° 33' 30''$.

3. $\cot^{-1} 2.1441$, $\tan^{-1} 0.4737$, $\tan^{-1} 1.7611$, $\cot^{-1} 1.7611$.
Ans. $25° 00' 15''$, $25° 20' 45''$, $60° 24' 40''$, $29° 35' 20''$.

4. Show that $\sin^{-1} 0.4250 + \sin^{-1} 0.9052 = 90°$.

5. Show that $\sin^{-1} 0.4488 + \sin^{-1} 0.4746 = 55°$.

6. If $\tan^{-1} 0.5000 + \tan^{-1} x = 87° 34'$, find x.

Ans. $x = 1.8040$.

7. In a right triangle one side $b = 4.63$ and the hypotenuse $c = 10.0$. Find the included angle A. *Ans.* $A = 62° 25'$.

(Suggestion. $\cos A = b/c$.)

8. In a right triangle the hypotenuse $c = 35.00$ and the side $a = 31.29$. Find the angle opposite a, and the third side b.

Ans. $A = 63° 23'$, $b = 15.68$.

9. Two sides of a right triangle are $a = 475.0$, $b = 237.5$. Find the angles A and B and the hypotenuse c.

Ans. $A = 63° 26'$, $B = 26° 34'$, $c = 531.1$.

10. Determine the uncertainty in the number of seconds of $\tan^{-1} 2.1211$ as given by a four-place table, assuming the principle of proportional parts. *Ans.* $\tan^{-1} 2.1211 = 64° 45' 30'' \pm 2''$.

20. Solution of Right Triangles by Natural Functions. In order to solve a right triangle, two parts must be given besides the right angle, and one of these parts must be a side.

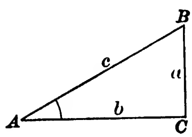


Fig. 23.

Let ABC be any right triangle, A, B, C the angles, and a, b, c the corresponding sides. Then

$$\frac{a}{c} = \sin A, \quad (1), \qquad \frac{b}{c} = \cos A, \quad (2), \qquad \frac{a}{b} = \tan A, \quad (3).$$

These three equations are sufficient for the solution of any right triangle, for,—

If one of the given parts is an angle, we may call this angle A, and the other given part must be either a, b, or c;

I. Given A and a; then (3) gives b, and (1) gives c.
II. Given A and b; then (2) gives c, and (3) gives a.
III. Given A and c; then (1) gives a, and (2) gives b.

If the given parts are both sides, there are three more cases, —

IV. Given a and b; then (3) gives A, and c is found as in I or II.
V. Given b and c; then (2) gives A, and a is found as in II or III.
VI. Given c and a; then (1) gives A, and B is found as in III or I.

Since A and B are complementary, B may always be found from the relation

$$A + B = 90°.$$

It is not necessary to consider the various cases of right triangles separately, for the following simple rule governs all cases:

Employ that trigonometric function of the angle which involves the two sides under consideration. The two sides may both be given, or one may be given and the other required to be found.

EXAMPLE 1. Given one side of a right triangle equal to 418.5, and the hypotenuse equal to 614.0; required the other parts.

Solution. Denote the given side by a or b, let us say by b, and the hypotenuse by c. We then have

Given $b = 418.5$,
$c = 614.0$.

Required $A = 47° \ 02'$,
$B = 42° \ 58'$,
$a = 449.3$.

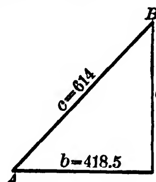


Fig. 24.

1. To find A.

We have
$$\cos A = \frac{b}{c} = \frac{418.5}{614.0} = 0.6816,$$

and from the table
$$A = \cos^{-1} 0.6816 = 47° \ 02'.$$

2. To find B.
$$A + B = 90°, \text{hence } B = 90° - 47° \ 02' = 42° \ 58'.$$

3. To find a.

From (3),
$$a = b \tan A.$$

From the table
$$\tan A = \tan 47° \ 02' = \quad 1.0736$$
$$b = \quad 418.5$$

Multiplying,
$$a = 449.3$$

4. Check. To guard against possible mistakes, the answers should be tested *by some formula which has not already been used* in obtaining the answers. Now in solving a right triangle we never use more than two of the three formulas (1), (2), (3), hence the third may always be used as a check. In the present problem, (2) was used in finding A, and (3) in finding a, hence we may test our results by using (1), that is, if our results are correct they should satisfy the relation (1), or

$$a = c \sin A$$
$$\sin A = \sin 47° \ 02' = \quad 0.7318$$
$$c = \quad 614$$

Multiplying, we get
$$a = 449.3,$$

which agrees with the value of a as determined above.

NOTE. We might have used the relation $a^2 + b^2 = c^2$ as a check, but this would have required more work.

EXAMPLE 2. Given one angle of a right triangle equal to $36^\circ 50'$, and the hypotenuse equal to 3.12; to solve the triangle.

Solution. Denote the angle by A, and the hypotenuse by c, then we have, —

Given $A = 36^\circ 50'$, $c = 3.12$.

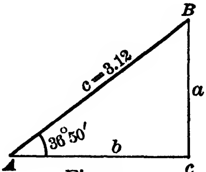


Fig. 25.

Required $a = 1.87$, $b = 2.50$, $B = 53^\circ 10'$.

1. To find a.

From (1), $a = c \sin A$.

From the table, $\sin A = \sin 36^\circ 50' = 0.5995$

$c = 3.12$

Multiplying, $a = 1.870$

2. To find b.

From (2), $b = c \cos A$.

From the tables, $\cos A = \cos 36^\circ 50' = 0.8004$

$c = 3.12$

Multiplying, $b = 2.497$

3. To find B.

$A + B = 90^\circ$, hence $B = 90^\circ - 36^\circ 50' = 53^\circ 10'$.

4. Check. Having already used (1) and (2), we use (3).

$a = b \tan A$.

From the tables, $\tan A = \tan 36^\circ 50' = 0.7490$

$b = 2.497$

Multiplying, we get $a = 1.870$,

which agrees with the value of a as obtained above.

Instead of referring to equations (1), (2), and (3), it is better to write down from the triangle under consideration that ratio which is needed to solve for a particular side or angle. The method will be sufficiently clear from an example.

EXAMPLE 3. Given $\phi = 15^\circ 25'$, $m = 345$.

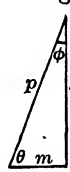


Fig. 26.

Required $\theta = 74^\circ 35'$, $n = 1251$, $p = 1298$.

Solution.

1. To find θ. $\theta + \phi = 90^\circ$,

$\theta = 90^\circ - 15^\circ 25' = 74^\circ 35'$.

2. To find n. Since ϕ and m are given, we must use that function of ϕ which involves n and the given side m.

$$\frac{m}{n} = \tan \phi, \quad \text{or} \quad n = m \cot \phi = 345 \times 3.6264 = 1251.1.$$

3. To find p. We use that function of ϕ which involves p and the given side m.

$$\frac{m}{p} = \sin\phi, \quad \text{or} \quad p = \frac{m}{\sin\phi} = \frac{345}{0.2658} = 1298.$$

4. Check. If our results for θ, n, and p are correct, we should have

$$\frac{n}{p} = \sin\theta, \quad \text{or} \quad n = p\sin\theta = 1298 \times 0.9640 = 1251.3.$$

In this case there is a slight discrepancy between the result of the check and the value of n as found in 2. This is due to the fact that we have given the value of n to five places while the table gives but four places. The values to four places agree exactly.

Exercise 13

Solve the following right triangles. Check your results when the answers are not given.

1. Given $a = 10.00$, $A = 25°$; find $b = 21.45$, $c = 23.66$.
2. Given $b = 256$, $A = 36°\ 30'$; find a, B, and c.
3. Given $c = 350$, $A = 56°45'$; find $B = 33°15'$, $a = 293$, $b = 192$.
4. Given $a = 346$, $B = 50°$; find the other parts.
5. Given $c = 45.7$, $B = 44°\ 50'$; find $a = 32.4$, $b = 32.2$.
6. Given $b = 13.5$, $B = 28°\ 40'$; solve the triangle.
7. Given $a = 170$, $b = 350$; find $A = 25°\ 54'$, $B = 64°\ 06'$, $c = 389$.
8. Given $a = 0.81$, $c = 2.54$; solve the triangle.
9. Given $b = 6.57$, $c = 10.6$; find $A = 51°\ 42'$, $B = 38°\ 18'$, $a = 8.32$.

10. Find the altitude of an isosceles triangle whose base is 368, and whose equal sides make an angle of 64°. *Ans.* 294.

11. Find the perimeter and area of a regular pentagon inscribed in a circle whose radius is 10. *Ans.* 58.78, 237.76.

12. Show that the area of any right triangle is equal to either one of the expressions $\frac{1}{2}\, bc \sin A$ or $\frac{1}{2}\, ac \sin B$.

13. In the right triangle ABC, Fig. 27, $AB = 338$, angle $B = 40°$; show how to find the length of the median AM, the length AS of the bisector of the angle A, and the angle included between these two.

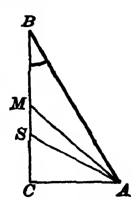


Fig. 27.

14. An oblique triangle ABC has $AB = 120$, $BC = 150$, and the angle $B = 67°30'$; solve the triangle.

Ans. $AC = 152$, $A = 65°41'$, $C = 46°49'$.

(Suggestion. Drop a perpendicular from A or C to the opposite side, dividing the triangle into two right triangles.)

15. Given one side of an oblique triangle equal to 57.3, and the adjacent angles equal to $35°45'$ and $75°30'$ respectively; find the remaining parts. *Ans.* 59.5, 35.9, $68°45'$.

(Suggestion. Draw the altitude from a vertex adjacent to the given side.)

21. Right Triangles Having a Small Angle.

Given the hypotenuse c and a side b of a right triangle, to solve the triangle in case b and c are nearly equal.

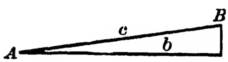


Fig. 28.

The angles A and B are given by the relation

$$\cos A = \frac{b}{c} = \sin B.$$

It is apparent from the figure that if b is nearly equal to c, angle A must be very small and angle B must be nearly equal to $90°$.

By examining the table of natural sines and cosines it will be seen that for small angles the cosines are so nearly equal that there is no difference at all in the first four places, and similarly the sines of angles near $90°$ are nearly equal. Thus, so far as the table shows, all angles from $0°$ to $0°\ 34'$ have the same cosine, and likewise all angles between $0°\ 34'$ and $0°\ 60'$, between $1°\ 00'$ and $1°\ 16'$, etc. It follows that *a small angle cannot be accurately found from its cosine nor an angle near 90° from its sine.*

To avoid using the cosine, the following formula is used whenever the given parts b and c are nearly equal, —

$$\tan\frac{A}{2} = \sqrt{\frac{c-b}{c+b}}.$$

Proof. Let ABC be any right triangle, b the base, a the altitude, c the hypotenuse. Produce CA to O, making $AO = c$. Join O and B. Now angle A ($= CAB$) $\doteq$ angle $AOB +$ angle ABO, hence since triangle AOB is isosceles, angle A = twice the angle AOB, that is, angle $AOB = \frac{A}{2}$, and

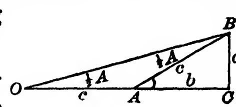


Fig. 29.

$$\tan\frac{A}{2} = \frac{a}{c+b} = \frac{\sqrt{c^2-b^2}}{c+b} = \sqrt{\frac{c^2-b^2}{(c+b)^2}} = \sqrt{\frac{c-b}{c+b}}.$$

EXAMPLE 1. Given $b = 25.7$, $c = 26.8$; to find A, B, and a.

Solution.
$$c - b = 26.8 - 25.7 = 1.1,$$
$$c + b = 26.8 + 25.7 = 52.5.$$
$$\tan\frac{A}{2} = \sqrt{\frac{1.1}{52.5}} = \sqrt{0.020952} = 0.1447,$$
$$\frac{A}{2} = 8^\circ\,14', \quad A = 16^\circ\,28',$$
$$B = 90^\circ - 16^\circ\,28' = 73^\circ\,32'.$$
$$a = b\tan A = 25.7 \times 0.2956 = 7.6.$$

Check.
$$a^2 = c^2 - b^2 = (c - b)(c + b).$$
$$a^2 = (7.6)^2 = 57.76,$$
$$(c - b)(c + b) = 1.1 \times 52.5 = 57.75.$$

EXERCISE 14

1. Given $b = 4.75$, $c = 5.25$; solve the triangle.
Ans. $A = 25^\circ\,13'$, $B = 64^\circ\,47'$, $a = 2.24$.

2. Given $a = 9.6$, $c = 10.4$; solve the triangle and check your results.

3. Show that $\tan\frac{A}{2} = \frac{c - b}{a}$.

4. From Fig. 29 show that $\sin\frac{A}{2} = \sqrt{\frac{c - b}{2c}}$, $\cos\frac{A}{2} = \sqrt{\frac{c + b}{2c}}$.

22. Historical Note. The use of tables of natural functions dates back to antiquity. In the second century B.C., Hipparchus, a Greek astronomer and mathematician, constructed tables of chords (double sines) which answered the same purpose as a table of sines. The tables of Hipparchus have been lost. The oldest table now extant is that of Ptolemaus (second century A.D.), giving double sines from minute to minute with an accuracy which in our system of numeration would be expressed by five places of decimals.

Hindu mathematicians as early as the fifth century A.D. were in possession of a small table which they memorized, very much as we memorize our multiplication table. Values not given in the table were computed from memory as occasion required, by means of a formula put in verse.

The first table which approached in extent and arrangement the tables now in use is the "Canon doctrinæ triangulorum" of Rheti-

cus (1551). This table gives each of the six functions for intervals of 10″ from 0 to 45. Like present tables, the degrees and seconds proceed from top to bottom in the left marginal columns, while the complementary angles proceed from bottom to top in the right marginal columns. Later, Rheticus prepared a second table for which the sines of angles for intervals of 10″ were computed to 15 places of decimals, though only 10 places were retained in the "Opus Palatinum," the name under which the table was published.

Rheticus' table contained numerous errors, which were largely removed by Pitiscus, an indefatigable arithmetician of the seventeenth century. In addition to revising the existing tables he computed anew the sines of angles from 0° to 7° for intervals of 1′ to from 20 to 25 places of decimals. Pitiscus's improved tables were published in 1613 under the title "Thesaurus mathematicus." These tables formed the basis of all subsequent tables, until the discovery of improved methods of computation in recent times has made it comparatively easy to check old tables or to compute new ones.

23. Review.

1. (*a*) Define trigonometry, explain the etymology of the word and tell how the science originated. (*b*) Define in words the sine, cosine, and tangent of an acute angle. (*c*) Define the secant, cosecant, and cotangent of an angle. (*d*) What is meant by the versine and coversine of an angle?

2. (*a*) Name three pairs of functions such that in each pair either is the cofunction of the other. (*b*) Explain the origin of the terms cosine, cosecant, and cotangent. (*c*) Prove that $\sin A = \cos (90° - A)$, $\cos A = \sin (90° - A)$, $\tan A = \cot (90° - A)$. (*d*) Express $\sin 76° 40'$ as a function of an angle less than 45°.

3. (*a*) Construct the following angles: $\sin^{-1} \frac{1}{3}$, $\cos^{-1} 0.4$, $\tan^{-1} 0.5$, $\cot^{-1} 3$. (*b*) Give from memory the values of the sine, cosine, and tangent of each of the following angles: 0°, 30°, 45°, 60°, 90°. (*c*) Draw a figure and deduce the functions of 30° and 45°.

4. (*a*) Name three pairs of functions such that in each pair either function is the reciprocal of the other. (*b*) Prove the relations $\sin^2 A + \cos^2 A = 1$, $\tan^2 A + 1 = \sec^2 A$, $\cot^2 A + 1 = \csc^2 A$, $\tan A = \dfrac{\sin A}{\cos A}$. (*c*) Given $\sin A = \dfrac{3}{5}$, find each of the other functions of A. (*d*) Express each of the functions of A in terms of $\tan A$.

5. (*a*) Reduce to its simplest form the expression

$$\frac{\cos X}{1 - \tan X} + \frac{\sin X}{1 - \cot X}.$$

(*b*) Prove the identities, —

$$\frac{\sec A + \csc A}{\sec A - \csc A} = \frac{\tan A + 1}{\tan A - 1} = \frac{1 + \cot A}{1 - \cot A}.$$

6. (*a*) What is meant by a table of natural functions? (*b*) Which functions increase and which decrease as the angle increases from 0° to 90°? (*c*) What is meant by the term "interpolation"? by the "principle of proportional parts"? (*d*) How would you find the secant of a given angle? (*e*) Can a small angle be found more accurately from its sine or from its cosine? Why? (*f*) Can a small angle be found more accurately from its tangent or from its cotangent? Why?

7. (*a*) There are four different cases of right-triangle problems according as the given parts are: I. The hypotenuse and an angle. II. One side and an angle. III. One side and the hypotenuse. IV. Two sides. Show how to solve each case.

8. Prove the formula $\tan\frac{A}{2} = \sqrt{\frac{c - b}{c + b}}$, and write down a similar formula for $\tan\frac{B}{2}$.

9. A flagstaff AB stands on top of a tower. To determine its height, a distance OP 500 feet long was measured off in a horizontal direction from the foot of the tower. At P the angles OPA and OPB were measured, and were found to be 22° and 28° respectively. Find the length of the flagstaff by natural functions, also by the graphic method, and compare your results.

CHAPTER IV

LOGARITHMS

24. Definition of Logarithm. The numbers representing the given parts of a triangle and other numbers obtained by careful measurement usually contain three or four significant figures and sometimes five, six, or it may be even seven significant figures. Multiplication, division, the extraction of roots and raising to powers of such numbers by the ordinary methods, require a great deal of tedious labor, most of which may be avoided by using another method of computation, known as the method of logarithms. This method requires the use of tables, by the aid of which multiplication of two or more numbers is accomplished by adding certain other numbers found in the tables. Similarly division is reduced to a mere subtraction of two numbers found in the tables. To raise to a power or to extract a root of a number, the corresponding number in the table is multiplied or divided by the index of the power or root.*

The method of logarithms presupposes that when some positive number, different from unity, has been chosen, every other positive number can be expressed as some power (integral, fractional, negative, or incommensurable) of this number. Thus, if a is some positive number not equal to one, and N any positive number, we assume that a number x can always be found such that

$$a^x = N.$$

This number x is called the *logarithm of* N to the *base* a, and is usually written

$$x = \log_a N,$$

in words:

* The usefulness of the method of logarithms may be anticipated from the testimony of Laplace, the great French astronomer, who said the method of logarithms "by reducing to a few days the labors of many months, doubles, as it were, the life of an astronomer, besides freeing him from the errors and disgust inseparable from long calculation." The advantages which the use of logarithms offers to the astronomer are shared, of course, by all others who deal much with numerical calculation.

The logarithm of a number to a given base is the exponent of the power to which the base must be raised to produce the number.*

For example, since

$$\begin{aligned}
10^2 &= 100, & 2 &= \log_{10} 100;\\
10^3 &= 1000, & 3 &= \log_{10} 1000;\\
10^{\frac{1}{2}} &= 3.1623, & \tfrac{1}{2} &= \log_{10} 3.1623;\\
10^{-1} &= 0.1, & -1 &= \log_{10} 0.1;\\
(0.5)^2 &= 0.25, & 2 &= \log_{0.5} 0.25;\\
27^{\frac{2}{3}} &= 9, & \tfrac{2}{3} &= \log_{27} 9;
\end{aligned}$$

$$2^6 = 4^3 = 8^2 = 64, \quad 6 = \log_2 64, \quad 3 = \log_4 64, \quad 2 = \log_8 64.$$

25. Fundamental Laws Governing Logarithms. Since logarithms are exponents, the laws of logarithms are the same as the laws of exponents. Now the laws of exponents are, —

(*a*) $a^x \cdot a^y = a^{x+y}$,

x is the logarithm (exponent) of the first factor,
y is the logarithm (exponent) of the second factor,
$x + y$ is the logarithm (exponent) of the product;

hence,

The logarithm of the product of two factors is equal to the sum of the logarithms of the factors, or

If $\quad P = M \cdot N, \qquad \log P = \log M + \log N.$

Similarly,

If $\quad P = L \cdot M \cdot N \cdot \cdots, \log P = \log L + \log M + \log N + \cdots.$

Thus $\log 15 = \log 3 + \log 5, \qquad \log 30 = \log 2 + \log 3 + \log 5.$

(*b*) $a^x \div a^y = a^{x-y}$,

x is the logarithm (exponent) of the dividend,
y is the logarithm (exponent) of the divisor,
$x - y$ is the logarithm (exponent) of the quotient;

hence,

* From *logos* = ratio, and *arithmos* = number, so called by the Scotch mathematician John Napier, one of the inventors of logarithms, because, as originally conceived of, logarithms were a set of numbers, as

$$h, \quad k, \quad l, \quad m, \quad n, \quad \text{etc.},$$

corresponding to a second set a^h, a^k, a^l, a^m, a^n, etc.,
the second set being so chosen that the ratio between any consecutive two is the same.

The logarithm of the quotient of two numbers is equal to the logarithm of the dividend diminished by the logarithm of the divisor, or,

If $\qquad Q = M \div N, \qquad \log Q = \log M - \log N.$

Thus $\qquad \log \frac{5}{3} = \log 5 - \log 3.$

(*c*) $(a^x)^n = a^{nx}$,

x is the logarithm of the quantity which is to be raised to the nth power, nx is the logarithm of the resulting power; hence,

The logarithm of any power of a number is equal to the logarithm of the number multiplied by the index of the power to which it is to be raised, or

If $\qquad P = N^n, \qquad \log P = n \log N.$

Thus $\qquad \log 3^5 = 5 \log 3.$

(*d*) $\sqrt[n]{a^x} = a^{\frac{x}{n}}$,

x is the logarithm of the quantity whose nth root is to be extracted, $\frac{x}{n}$ is the logarithm of the resulting root; hence,

The logarithm of any root of a number is equal to the logarithm of the number divided by the index of the root to be extracted, or

If $\qquad P = \sqrt[n]{N}, \qquad \log P = \dfrac{\log N}{n}.$

Thus $\qquad \log \sqrt[5]{17} = \frac{1}{5} \log 17.$

26. Logarithms of Special Values.

(*a*) $a^1 = a$, hence $\log a = 1$, that is,

The logarithm of the base is 1.

(*b*) Any number divided by itself is 1, but by the law of exponents $a \div a = a^0$, so that $a^0 = 1$, and therefore $\log_a 1 = 0$, that is,

The logarithm of 1 *to any base is* 0.

(*c*) By Art. 25 (*b*), $\log \frac{1}{N} = \log 1 - \log N$, and by (*b*) of this article, $\log 1 = 0$, hence

$$\log \frac{1}{N} = -\log N,$$

that is,

The logarithm of the reciprocal of any number is equal to minus the logarithm of the number.

Definition. *The logarithm of the reciprocal of a number is called the* COLOGARITHM * *of the number.*

Thus

$\log_{10} 10 = 1,$ hence $\log_{10} \frac{1}{10} = -1 = \text{colog}_{10} 10;$

$\log_{10} 100 = 2,$ hence $\log_{10} \frac{1}{100} = -2 = \text{colog}_{10} 100;$

$\log_{10} 3.1623 = \frac{1}{2},$ hence $\log_{10} \dfrac{1}{3.1623} = -\frac{1}{2} = \text{colog}_{10} 3.1623.$

Since $\dfrac{M}{N} = M \cdot \dfrac{1}{N}$, we have

$$\log \frac{M}{N} = \log M + \log \frac{1}{N} = \log M + \text{colog } N,$$

that is,

The logarithm of the quotient of two numbers is equal to the logarithm of the dividend plus the cologarithm of the divisor.

EXERCISE 15

1. Given $5^3 = 125$, $5^2 = 25$, $5^1 = 5$; write down the logarithms to the base 5, of 125, 25, 5, $\frac{1}{5}$, $\frac{1}{25}$, $\frac{1}{125}$.

2. Given $\log_4 2 = 0.5$, $\log_3 81 = 4$, $\log_{10} 3.1623 = 0.5$, $\log_b a = c$; write down equivalent expressions free from the symbol log.

Ans. $4^{0.5} = 2$, $3^4 = 81$, etc.

3. Find $\log_3 27$, $\log_{10} 10{,}000$, $\log_a a$, $\log_3 \frac{1}{27}$, $\log_a a^{\frac{3}{2}}$.

Ans. 3, 4, 1, -3, $\frac{3}{2}$.

4. Express in terms of log 2, log 3, and log 5, the following:

$$\log 15,\ \log \frac{10}{3},\ \log \frac{5}{6},\ \log 100,\ \log \sqrt{\frac{15}{2}},\ \log \frac{\sqrt{6}\sqrt[3]{15}}{\sqrt[5]{60}}.$$

Ans. $\log 3 + \log 5$, $\log 2 + \log 5 - \log 3$, $\log 5 - \log 2 - \log 3$, $2(\log 2 + \log 5)$, $\frac{1}{2}(\log 3 + \log 5 - \log 2)$, $\frac{1}{10} \log 2 + \frac{19}{30} \log 3 + \frac{2}{15} \log 5$.

5. Given $\log_{10} 2 = 0.30103$, $\log_{10} 3 = 0.47712$, $\log_{10} 5 = 0.69897$; find $\log_{10} 4$, $\log_{10} 0.3$, $\log_{10} 0.75$, $\log_{10} \sqrt[3]{5}$, $\log_{10} \sqrt{\dfrac{1280}{729}}$.

Ans. 0.60206, $-1 + 0.47712$,† $-1 + 0.87506$, 0.23299, 0.12224.

* Whenever $x + y =$ constant, x and y are said to be complements (more specifically arithmetic complements) of each other. Now $\log x + \log \frac{1}{x} = \log 1$ = constant, hence $\log \frac{1}{x}$ is the complement of $\log x$, which on contraction becomes colog x.

† When 10 is the base, the logarithm is always written so that the fractional part is positive.

6. If $b = 184$, $c = 59$; find the logarithm of $\sqrt{\dfrac{b+c}{b-c}}$, and also of $b^2 - c^2$.
Ans. 0.14434, 4.48251.

7. Show that the fractional part of a logarithm, to the base 10, is not changed if the number is multiplied by 10, or by a power of 10.

8. Compute to four places of decimals the numbers whose logarithms to the base 10 are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{6}$.
Ans. 3.1623, 1.7783, 2.1544, 1.4678.

27. The Common System of Logarithms. Each different base determines a different system of logarithms. The system in common use has 10 for its base, and is called the *common* * *system* of logarithms. Common logarithms have been carefully computed and tabulated, so that the logarithm of any number can be readily found by referring to the table, and, vice versa, if the logarithm of a number is known, the number itself can be found from the table.

The advantage of the common system over other systems of logarithms consists in this: the fractional part of any common logarithm remains unchanged when the number is multiplied or divided by 10 or a power of 10. To see this, let us consider a special case, say the number whose logarithm is 0.5.

$$10^{0.5} = 10^{\frac{1}{2}} = \sqrt{10} = 3.16228, \quad \text{therefore } \log\dagger\, 3.16228 = 0.5.$$

Multiplying by 10,

$$10 \times 10^{0.5} = 10 \times 3.16228,$$

or

$$10^{1.5} = 31.6228, \quad \text{therefore } \log 31.6228 = 1.5.$$

Similarly

$$10^{2.5} = 316.228, \quad \text{therefore } \log 316.228 = 2.5;$$
$$10^{3.5} = 3162.28, \quad \text{therefore } \log 3162.28 = 3.5;$$
$$10^{4.5} = 31622.8, \quad \text{therefore } \log 31622.8 = 4.5; \quad \text{etc.}$$

Again, dividing by 10, we have,

$$\frac{10^{0.5}}{10} = 10^{-1} \times 10^{0.5} = \frac{3.16228}{10}$$

or

$$10^{-1+0.5} = 0.316228, \qquad \log 0.316228 = -1 + 0.5 = 9.5 - 10.$$

* This system is also known as the *Briggsian system*, in honor of Henry Briggs of Oxford (1556–1630), who was the first one to compute and publish a table of logarithms to the base 10.

† In the remainder of this chapter, and always when computations are concerned, if no base is expressed, the base 10 is understood.

Similarly

$$10^{-2+0.5} = 0.0316228, \quad \log 0.0316228 = -2 + 0.5 = 8.5 - 10;$$
$$10^{-3+0.5} = 0.00316228, \quad \log 0.00316228 = -3 + 0.5 = 7.5 - 10;$$

and so on, the form $9.5 - 10$ being introduced to avoid the plus sign between -1 and 0.5.

Now the logarithms of each of the numbers 3.16228, 31.6228, 316.228, 3162.28, 31622.8, etc., 0.316228, 0.0316228, 0.00316228, etc., have the same fractional part, namely, 0.5, while the integral parts 1, 2, 3, 4, etc., -1, -2, -3, etc., plainly depend upon the position of the decimal point.

For convenience the integral part of the logarithm is called the *characteristic* of the logarithm, and the fractional part is called the *mantissa.*

Thus, the logarithm of 316.228 is composed of the characteristic 2 and the mantissa 0.5.

The logarithm of 0.00316228 is composed of the characteristic -3 and the mantissa 0.5.

We have seen that the mantissa is independent of the position of the decimal point, that is, it is the same for all numbers composed of the same figures taken in the same order.

28. Rule for the Characteristic. We shall now learn how the characteristic of a logarithm of a number may be determined before the logarithm is known.

$$\begin{aligned} 10^0 &= 1, & &\text{or} & \log 1 &= 0; \\ 10^1 &= 10, & &\text{or} & \log 10 &= 1; \\ 10^2 &= 100, & &\text{or} & \log 100 &= 2; \\ 10^3 &= 1000, & &\text{or} & \log 1000 &= 3; \quad \text{etc.} \end{aligned}$$

Since $\log 1 = 0$ and $\log 10 = 1$,

every number between 1 and 10 has a logarithm between 0 and 1,* or, every number whose integral part has *one digit* has a logarithm whose characteristic is *zero.*

Since $\log 10 = 1$ and $\log 100 = 2$,

every number between 10 and 100 has a logarithm between 1 and 2,

* This statement assumes that to the greater of two numbers corresponds the greater logarithm, that is, if $M > N$, $\log M > \log N$, a theorem which can be easily proven by elementary algebra.

or, every number whose integral part has *two digits* has a logarithm whose characteristic is *one.*

Similarly, every number between 100 and 1000 has a logarithm between 2 and 3, or, every number whose integral part has *three digits* has a logarithm whose characteristic is *two.*

Every number whose integral part has *four digits* has a logarithm whose characteristic is *three,* and so on.

This gives us the following rule:

I. *The characteristic of the logarithm of any number greater than 1 is one less than the number of digits in the integral part of the number.*

Thus, the integral part of the number 31622.8 consists of 5 digits, hence the characteristic of its logarithm is 4.

The integral part of 3.16228 consists of 1 digit, hence the characteristic of its logarithm is 0.

Again,

$$10^0 = 1, \qquad \text{or } \log 1 = 0;$$

$$10^{-1} = \frac{1}{10} = 0.1, \qquad \text{or } \log 0.1 = -1;$$

$$10^{-2} = \frac{1}{10^2} = 0.01, \qquad \text{or } \log 0.01 = -2;$$

$$10^{-3} = \frac{1}{10^3} = 0.001, \qquad \text{or } \log 0.001 = -3;$$

$$10^{-4} = \frac{1}{10^4} = 0.0001, \qquad \text{or } \log 0.0001 = -4; \text{etc.}$$

Hence, every number between 1 and 0.1 has a logarithm between 0 and -1, or, every fraction greater than 0.1 has a logarithm whose characteristic is -1.

Similarly, every number between 0.1 and 0.01 has a logarithm between -1 and -2, or, every fraction greater than 0.01 but less than 0.1 has a logarithm whose characteristic is -2.

In like manner, every fraction greater than 0.001 but less than 0.01 has a logarithm whose characteristic is -3, and so on.

This gives us a second rule:

II. *The characteristic of the logarithm of any number less than 1 is a negative number one more than the number of ciphers between the decimal point and the first significant figure of the fraction.*

Thus, the fraction 0.00316228 has two ciphers between the decimal point and the first significant figure, hence the characteristic of its logarithm is -3 or $7 - 10$.

The fraction 0.316228 has no cipher between the decimal point and the first significant figure, hence the characteristic of its logarithm is -1 or $9 - 10$.

Exercise 16

1. Write down the characteristics of the common logarithms of 367, 36.7, 3670, 3.67, 0.000367, 0.367.

Ans. 2, 1, 3, 0, -4 or $6 - 10$, -1 or $9 - 10$.

2. $\log 635 = 2.80277$; write down the logarithms of 6.35, 63500, 0.635, 0.0000635.

Ans. 0.80277, 4.80277, $9.80277 - 10$, $5.80277 - 10$.

3. How many digits are there in the integral part of the number whose logarithm is 3.1567; 1.6533; 0.6831?

4. To the number 57 corresponds the mantissa 75587; what is the number whose logarithm is 1.75587; 2.75587; 0.75587; $9.75587 - 10$?

Ans. 57; 570; 5.7; 0.57.

5. To the number 673 corresponds the mantissa 82802; find the logarithm of 673; of $(673)^2$; of $\sqrt{673}$; of $\sqrt[3]{(673)^2}$.

Ans. 2.82802; 5.65604; 1.41401; 1.88535.

6. $\log 3 = 0.47712$; how many digits are there in 3^{25}; in 3^{100}; in 30^{15}; in 27^5?

Ans. 12; 48; 23; 8.

7. Write down the cologarithms of the numbers in problem 2.

Ans. $9.19723 - 10$; $5.19723 - 10$; 0.19723; 4.19723.

8. $\log 5 = 0.69897$; find the logarithm of $\frac{1}{5}$; of $\frac{1}{25}$; of 0.05; of $\sqrt{5}$; of $\sqrt{\frac{1}{5}}$; of $\sqrt[3]{0.5}$; of $\sqrt[3]{\frac{1}{5}}$.

Ans. $9.30103 - 10$; $8.60206 - 10$; $8.69897 - 10$; 0.34948; $9.65052 - 10$; $9.89966 - 10$; $9.76701 - 10$.

9. To the number 3 corresponds the mantissa 47712, and to the number 7 corresponds the mantissa 84510; find the logarithm of 21; of $\frac{3}{7}$; of $\sqrt{0.3 \times 49}$; of $\sqrt[3]{\frac{7}{9}}$.

Ans. 1.32222; $9.63202 - 10$; 0.58366; $9.96362 - 10$.

10. The formula for the amount (A) of a principal (P) put out on compound interest at (R) per cent for (t) years, the interest being compounded annually, is

$$A = P\left(1 + \frac{R}{100}\right)^t,$$

whence

$$\log A = \log \mathrm{P} + t \log\left(1 + \frac{R}{100}\right).$$

Find the number of digits in the amount of \$1 at compound interest at 6 per cent for 100 years, the mantissa corresponding to 106 being .02531. *Ans.* 3 digits.

29. Tables of Common Logarithms. Our next step is to learn how to use a table of logarithms in actual computation.

Common logarithms, except those belonging to numbers which are integral powers of 10, cannot be exactly expressed in decimals. We must therefore omit all the figures after a certain decimal place. Just where to stop depends upon the accuracy desired. If some of the numbers which enter a given problem are the results of measurement, as is frequently the case, their accuracy will not ordinarily exceed four or five figures, consequently it will be useless to retain more than five figures in the decimal part of their logarithms. In other words, a table of logarithms which contains the mantissas to five places will answer for the solution of most practical problems in which approximate answers are all that is necessary or possible. Such a table is known as a five-place table of logarithms.

The explanations which follow, and all the answers in this book obtained from logarithmic computation, are based upon a five-place table.

When a five-place table is used, it is not worth while to retain more than five significant figures in the answers to the problems, and even then the last figure is not always exact.

When more than five-place accuracy is required, and even when the fifth place must be known with certainty, larger tables must be used. Tables containing six places, seven places, eight places, ten places, eleven places, twenty places, and partial tables containing up to two hundred and sixty places, have been published. There are also smaller tables containing three and four places only.

Tables containing more than seven places are seldom used, for seven-

place tables meet practically every demand of present-day science. The results obtained by means of a seven-place table are as exact as the most careful measurements obtained by the most skillful observers by means of the most precise instruments under the most favorable conditions.

30. To Find the Logarithm of a Given Number.

(*a*) *When the given number has four figures.* The characteristic is found by the rule in Art. 28. To find the mantissa, enter that line of the table which begins with the number made up of the first three figures of the given number and take out that number of the line which is found in the column headed by the fourth figure of the given number.

The number thus found constitutes the third, fourth and fifth figures of the required mantissa. The first and second figures are found in the column headed by 0 either in or above the line in which the third, fourth and fifth figures are found, except when these figures as given in the table are preceded by a star (*) in which case the first two figures are found in the next following line.

EXAMPLE 1. From the specimen page of logarithms, page 63, we find

mantissa log 6315 = 80037, hence log 6315 = 3.80037;
mantissa log 65.24 = 81451, hence log 65.24 = 1.81451;
mantissa log 6.608 = 82007, hence log 6.608 = 0.82007.

(*b*) *When the given number has less than four figures.* We know that the mantissa of the logarithm of a number is not changed if the number is multiplied or divided by 10 or by some power of 10. After we have determined the characteristic of a logarithm we may then annex as many ciphers to the given number as we need to make up four places and find the mantissa of this new number by case (*a*).

EXAMPLE 2. Find the logarithm of 64.

The characteristic of log 64 is 1. The mantissa of log 64 is the same as the mantissa of log 6400, which by case (*a*) is found to be 80618. Hence,

$$\log 64 = 1.80618.$$

(*c*) *When the number has more than four figures.*

EXAMPLE 3. Find the logarithm of 6425.4.

The characteristic of log 6425.4 = 3.

(SPECIMEN PAGE)

N	0	1	2	3	4	5	6	7	8	9	D
630	934	941	948	955	962	969	975	982	989	996	7
631	80 003	010	017	024	030	037	044	051	058	065	7
632	072	079	085	092	099	106	113	120	127	134	7
633	140	147	154	161	168	175	182	188	195	202	7
634	209	216	223	229	236	243	250	257	264	271	7
635	277	284	291	298	305	312	318	325	332	339	7
636	346	353	359	366	373	380	387	393	400	407	7
637	414	421	428	434	441	448	455	462	468	475	7
638	482	489	496	502	509	516	523	530	536	543	7
639	550	557	564	570	577	584	591	598	604	611	7
640	618	625	632	638	645	652	659	665	672	679	7
641	686	693	699	706	713	720	726	733	740	747	7
642	754	760	767	774	781	787	794	801	808	814	7
643	821	828	835	841	848	855	862	868	875	882	7
644	889	895	902	909	916	922	929	936	943	949	7
645	956	963	969	976	983	990	996	*003	*010	*017	7
646	81 023	030	037	043	050	057	064	070	077	084	7
647	090	097	104	111	117	124	131	137	144	151	7
648	158	164	171	178	184	191	198	204	211	218	7
649	224	231	238	245	251	258	265	271	278	285	7
650	291	298	305	311	318	325	331	338	345	351	7
651	358	365	371	378	385	391	398	405	411	418	7
652	425	431	438	445	451	458	465	471	478	485	7
653	491	498	505	511	518	525	531	538	544	551	7
654	558	564	571	578	584	591	598	604	611	617	7
655	624	631	637	644	651	657	664	671	677	684	7
656	690	697	704	710	717	723	730	737	743	750	7
657	757	763	770	776	783	790	796	803	809	816	7
658	823	829	836	842	849	856	862	869	875	882	7
659	889	895	902	908	915	921	928	935	941	948	7
660	954	961	968	974	981	987	994	*000	*007	*014	7
661	82 020	027	033	040	046	053	060	066	073	079	7
662	086	092	099	105	112	119	125	132	138	145	7
663	151	158	164	171	178	184	191	197	204	210	7
664	217	223	230	236	243	249	256	263	269	276	7
665	282	289	295	302	308	315	321	328	334	341	7
666	347	354	360	367	373	380	387	393	400	406	7
667	413	419	426	432	439	445	452	458	465	471	7
668	478	484	491	497	504	510	517	523	530	536	7
669	543	549	556	562	569	575	582	588	595	601	7
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**D**

$$\begin{array}{rl} \text{By } (a), \text{ the mantissa of } \log 6425 = & 80787 \\ \text{the mantissa of } \log 6426 = & 80794 \\ \hline \text{difference for } 1 = & 7 \end{array}$$

Now the mantissa of log 6425.4 is evidently larger than 80787 and less than 80794, and since 6425.4 lies $\frac{4}{10}$ the way between 6425 and 6426, we will assume that the mantissa of log 6425.4 lies $\frac{4}{10}$ the way between 80787 and 80794. We must therefore increase the smaller of the two mantissas by $\frac{4}{10}$ of the difference between the two, that is, by $\frac{4}{10}$ of 7.

$$\tfrac{4}{10} \text{ of } 7 = 2.8 \text{ or } 3 \text{ to the nearest integer,}$$

and the mantissa of log $6425.4 = 80787 + 3 = 80790$.
Hence

$$\log 6425.4 = 3.80790.$$

We have here assumed the *principle of proportional parts for logarithms of numbers*, namely, that for small changes in the number, the change in the logarithm is proportional to the change in the number.

EXAMPLE 4. Find the logarithm of 6.5487.

The characteristic of log $6.5487 = 0$. The mantissa of log 6.5487 is the same as the mantissa of log 6548.7.

$$\begin{array}{rl} \text{mantissa } \log 6548 = & 81611 \\ \text{mantissa } \log 6549 = & 81617 \\ \hline \text{difference for } 1 = & 6 \end{array}$$

difference for $0.7 = 0.7$ of $6 = 4.2$ or 4 to the nearest integer.

Hence mantissa log $6548.7 = 81611 + 4 = 81615$,

and $\log 6.5487 = 0.81615$.

EXAMPLE 5. Find log 0.000635945.

The characteristic of log $0.000635945 = -4$ or $6 - 10$.

The mantissa of log 0.000635945 is the same as the mantissa of log 6359.45.

$$\begin{array}{rl} \text{mantissa } \log 6359 = & 80339 \\ \text{mantissa } \log 6360 = & 80346 \\ \hline \text{difference for } 1 = & 7 \end{array}$$

difference for $0.45 = 0.45$ of $7 = 3.15$ or 3 to the nearest integer.

Hence mantissa log $6359.45 = 80339 + 3 = 80342$,

and $\log 0.000635945 = 6.80342 - 10$.

EXAMPLE 6. Find the cologarithm of 65.021.

By definition, Art. 26, the cologarithm of a number is the logarithm of the reciprocal of that number, hence

$$\operatorname{colog} 65.021 = \log 1 - \log 65.021.$$

$$\begin{array}{rl} \log 1 = 0 = & 10 \qquad\quad - 10 \\ \underline{\log 65.021 =} & \underline{1.81306 \qquad\quad} \\ \operatorname{colog} 65.021 = & 8.18694^{*} - 10. \end{array}$$

31. To Find the Number Corresponding to a Given Logarithm.

(*a*) *When the given mantissa can be found in the table.* The characteristic is used only to determine the position of the decimal point after the number has been found. Find the given mantissa in the table. The first three figures of the number sought are found in the same line with the mantissa, in the column on the left, and the fourth figure is found at the top of the column containing the given mantissa.

EXAMPLE 1. Find the number whose logarithm is 5.82158.

Find the given mantissa 82158 in the table (see specimen page, p. 63). The number in the left-hand column and in the same line with the given mantissa is 663, and the number at the top of the column containing the mantissa 82158 is 1, hence the significant figures of the required number are 6631. The characteristic is 5, hence the integral part of the required number has 6 places, that is, the required number is 663100.

EXAMPLE 2. Find the number whose logarithm is 8.81043 − 10.

Corresponding to the mantissa 81043 we find in the table the number 6463. The characteristic is 8 − 10 or − 2, hence by the rule for the characteristic the required number is a decimal fraction with one cipher preceding the first significant figure.

Therefore the required number is 0.06463.

(*b*) *When the given mantissa cannot be found in the table*, two other consecutive mantissas can always be found in the table, one of which is a little smaller and the other a little larger than the given mantissa. The four figures corresponding to the smaller of these mantissas will be the first four figures of the required number; the fifth

* The subtraction is performed from left to right by subtracting each figure from 9 except the last one, which is subtracted from 10, thus: 1 from 9 = 8, 8 from 9 = 1, 1 from 9 = 8, 3 from 9 = 6, 0 from 9 = 9, 6 from 10 = 4.

figure, and sometimes the sixth,* can then be found by interpolation from the principle of proportional parts.

EXAMPLE 3. $\log N = 1.80395$, to find N.

The table does not contain the mantissa 80395, but it contains the two consecutive mantissas 80393 and 80400, one of which is smaller and the other larger than the given mantissa. The numbers corresponding to these mantissas are 6367 and 6368 respectively.

$$\begin{aligned} \text{mantissa } \log 6367 &= 80393 \\ \text{mantissa } \log 6368 &= 80400 \\ \hline \text{difference for } 1 &= \quad\quad 7 \end{aligned}$$

Since the given mantissa lies between 80393 and 80400, we infer that the number corresponding to the given mantissa lies between 6367 and 6368; let it be denoted by $6367 + x$, we then have

$$\begin{aligned} \text{mantissa } \log 6367 &= 80393 \\ \text{mantissa } \log 6367 + x &= 80395 \\ \hline \text{difference for } x &= \quad\quad 2 \end{aligned}$$

and the principle of proportional parts gives

$$1 : 7 = x : 2, \text{ that is, } x = \tfrac{2}{7},$$

and the number corresponding to the given mantissa is $6367\tfrac{2}{7}$. It still remains to determine the decimal point. The characteristic is 1, hence the required number is

$$N = 63.67\tfrac{2}{7} = 63.673 \text{ to five figures.}$$

EXERCISE 17

(In this exercise the specimen page of logarithms may be used.)

1. Find the logarithms of the following numbers: 6315, 632.5 6.454, 0.0655, 0.0065.

Ans. 3.80037, 2.80106, 0.80983, 8.81624 − 10, 7.81291 − 10.

2. Find the cologarithms of each of the numbers in 1.

Ans. 6.19963 − 10, 7.19894 − 10, 9.19017 − 10, 1.18376, 2.18709.

* In the first part of the table, the sixth significant figure of a number may be found by interpolation. In the latter part of the table, where the difference in the mantissas corresponding to a difference of 1 in the numbers is much smaller than in the first part of the table, the sixth figure of the number obtained by the principle of proportional parts cannot be depended on.

3. Find log 63.454, log 65.061, log 6.6095, log 0.0064159.

Ans. 1.80246, 1.81332, 0.82017, 7.80725 − 10.

4. Find the numbers whose logarithms are: 1.80277, 2.80584, 0.81003, 9.81351 − 10, 8.80017 − 10, 3.81184.

Ans. 63.50, 639.5, 6.457, 0.6509, 0.06312, 6484.

5. Find the numbers whose logarithms are: 1.80958, 2.81922, 0.81006, 9.80002 − 10, 8.80022 − 10, 3.82506.

Ans. 64.503, 659.51, 6.4574, 0.63099, 0.063127, 6684.3.

6. log 63.275 = x, log y = 1.81864; find x and y.

Ans. $x = 1.80124$, $y = 65.863$.

7. Without multiplying or dividing the numbers, find the logarithms of 66.027×0.65034, $6301 \div 6.454$, $(6535.4)^2$, $\sqrt{63.275}$.

Ans. 1.63287, 2.98958, 7.63055, 0.90062.

8. $N = \sqrt[6]{64550}$; find log N and then N.

Ans. $\log N = 0.80165$, $N = 6.3336$.

9. Find $(6.3096)^5$ by means of logarithms. *Ans.* 10000.

10. Find the logarithms of sin 40°, cos 48° 30′, cot 8° 50′.

Ans. 9.80808 − 10, 9.82125 − 10, 0.80854.

(Suggestion. First find the natural functions of the given angles and then find the logarithms of the resulting numbers.)

11. Find the first three significant figures and the number of figures in the integral part of the twenty-fifth power of 6.2.

Ans. 645, 20 places.

32. Directions for the Use of Logarithms. The following directions will aid the student in an intelligent use of logarithmic tables.

(*a*) In finding the logarithm of a given number, or in finding the number corresponding to a given logarithm, the interpolation should be performed mentally and only the complete result set down in writing.

(*b*) In writing down the cologarithm of a number, the subtraction from 10 should be performed mentally and from left to right.

(*c*) The results obtained by logarithms are approximations only. By neglecting the sixth and following significant figures of a number the inaccuracy introduced can never exceed one-half a unit in the

fifth place, that is, the error cannot exceed the $\frac{1}{200}$ part of 1 per cent.

(*d*) When the sixth figure of a number is 5, and we wish to retain only five significant figures, it is immaterial whether we increase the fifth figure by 1 or leave it unchanged. In case we increase the fifth figure by 1 the resulting number will be too large by half a unit in the fifth place; if we leave the fifth figure unchanged, the number will be too small by half a unit. In order to cause the inaccuracies arising from this source to offset one another, it is customary, on dropping a final 5, to increase the preceding figure by 1 when it is odd, but to leave it unchanged when it is even.

Thus,	0.154755 becomes 0.15476,
but	0.154745 becomes 0.15474;
one-half of	4.23453 becomes 2.11726,
but one-half of	4.23463 becomes 2.11732,

when the results are abridged to five places.

(*e*) Sometimes the nature of the problem is such that a four-place table would give all the accuracy required. In that case the fifth figure of the mantissa in the table may be omitted and the fourth figure increased by 1 if the omitted figure exceeds 5. If the final figure of the mantissa given in the table is marked with a stroke, thus $\bar{5}$, on omitting it the preceding figure is left unchanged; but if the final 5 is unmarked, the preceding figure is increased by 1. The reason for this is that the final 5 of a mantissa is itself the result of approximation, that is, it is either in defect (5 plus something less than $\frac{1}{2}$), or it is in excess (4 plus something greater than $\frac{1}{2}$), and the latter case is distinguished from the former by printing a stroke over the 5.

Thus from the table,

$$\log 2.078 = 0.3176\bar{5} = 0.3176,$$

but

$$\log 2.079 = 0.31785 = 0.3179,$$

when abridged to four places.

(*f*) Every logarithm consists of two parts,—the mantissa, which is always positive, and the characteristic, which is always integral (or 0), but may be negative as well as positive. When the characteristic is negative, it is customary to change the form of the logarithm by adding and subtracting 10.

Thus, $\log 0.4562 = -1 + 0.65916$ is written $9.65916 - 10$,

$\log 0.0032 = -3 + 0.5051\overline{5}$ is written $7.5051\overline{5} - 10$.

This is done in part to avoid mistakes which might arise from confusing the positive and the negative parts of logarithms.

For similar reasons, if a logarithm whose characteristic is negative is to be divided by 2, as in extracting a square root, we first modify its form by adding and subtracting 20; if the logarithm is to be divided by 3, we first add and subtract 30; and generally, if the logarithm whose characteristic is negative is to be divided by n, we first modify its form by adding and subtracting $10n$.

Thus,
$$\log 0.03254 = -2 + 0.51242 = 8.51242 - 10,$$
$$\tfrac{1}{2} \text{ of } \log 0.03254 = \tfrac{1}{2} \text{ of } (18.51242 - 20) = 9.25621 - 10,$$
$$\tfrac{1}{3} \text{ of } \log 0.03254 = \tfrac{1}{3} \text{ of } (28.51242 - 30) = 9.50414 - 10,$$
$$\tfrac{1}{4} \text{ of } \log 0.03254 = \tfrac{1}{4} \text{ of } (38.51242 - 40) = 9.62811 - 10,$$
$$\tfrac{1}{5} \text{ of } \log 0.03254 = \tfrac{1}{5} \text{ of } (48.51242 - 50) = 9.70248 - 10, \text{ etc.}$$

(*g*) The difference between two consecutive mantissas is called the *tabular difference* and is printed under D in the last column on each page of logarithms. At the bottom of each of the first three pages of logarithms the tabular differences which occur on the page are multiplied by each of the nine digits expressed as tenths. The resulting tables, known as *tables of proportional parts*, are used as an aid in interpolation.

EXAMPLE 1. Find by logarithms the product of 37.543 by 0.85734.

Solution. Denote the product by x.

From table I, $\log 37.543 = 1.57453$

$\log 0.85734 = 9.93315 - 10$

By Art. 25, (*a*), $\log x = 1.50768$

By Art. 31, and the table, $x = 32.187.$

EXAMPLE 2. Find by logarithms the quotient of 6.3725 divided by 82.756.

Solution. Denote the quotient by x.

From the table, $\log 6.3725 = 0.80431$

By Art 26, (*c*), and the table, $\operatorname{colog} 82.756 = 8.08220 - 10$

$\log x = 8.88651 - 10$

From the table, $x = 0.077003.$

EXAMPLE 3. Find the square root of 0.89355.

$$
\begin{array}{lrl}
\text{Solution.} & \log 0.89355 = & 9.95112 - 10 \\
 & = & 19.95112 - 20 \\
\text{Dividing by 2,} & \log \sqrt{0.89355} = & 9.97556 - 10 \\
\text{Hence} & \sqrt{0.89355} = & 0.94528.
\end{array}
$$

EXAMPLE 4. Find x, if $x^3 = (0.5824)^2$.

$$
\begin{array}{lrl}
\text{Solution.} & \log 0.5824 = & 9.76522 - 10 \\
\text{Multiplying by 2,} & \log x^3 = & 19.53044 - 20 \\
 & = & 29.53044 - 30 \\
\text{Dividing by 3,} & \log x = & 9.84348 - 10 \\
 & x = & 0.69740.
\end{array}
$$

EXAMPLE 5. $x = \dfrac{8.35 \times (62.5)^2 \times \sqrt{5.673}}{(1.256)^3 \times 623.7 \times \sqrt[3]{5.736}}$; find x.

Solution. By the rules of Art. 25,

$$
\begin{aligned}
\log x = {} & \log 8.35 + 2 \log 62.5 + \tfrac{1}{2} \log 5.673 \\
& + 3 \operatorname{colog} 1.256 + \operatorname{colog} 623.7 + \tfrac{1}{3} \operatorname{colog} 5.736.
\end{aligned}
$$

$$
\begin{array}{rll}
\log 8.35 & & = 0.92169 \\
\log 62.5 & = 1.79588 & \\
2 \log 62.5 & = 2 \times 1.79588 & = 3.59176 \\
\log 5.673 & = 0.75381 & \\
\tfrac{1}{2} \log 5.673 & = \tfrac{1}{2} \times 0.75381 & = 0.37690 \\
\operatorname{colog} 1.256 & = 9.90101 - 10 & \\
3 \operatorname{colog} 1.256 & = 3 \times (9.90101 - 10) & = 9.70303 - 10 \\
\operatorname{colog} 623.7 & & = 7.20502 - 10 \\
\operatorname{colog} 5.736 & = 9.24139 - 10 & \\
\tfrac{1}{3} \operatorname{colog} 5.736 & = \tfrac{1}{3} \times (29.24139 - 30) & = \underline{9.74713 - 10} \\
 & \log x & = 31.54553 - 30 \\
 & & = 1.54553 \\
 & x & = 35.118.
\end{array}
$$

EXERCISE 18

Solve by logarithms:

1. 3.784×7.843. *Ans.* 29.677.
2. 67.845×0.03457.
3. $0.67375 \div 3.468$. *Ans.* 0.19428.
4. $92.57 \div 1.3785$.

5. $(684.7)^2 \times (0.03873)^3$. *Ans.* 27.236.

6. $(0.8003)^3 \div \sqrt{5.73}$.

7. $\sqrt[5]{3284.5}$. *Ans.* 5.05.

8. $(0.12562)^{10}$.

9. $\dfrac{1.56 \times 37.8 \times \sqrt{0.0753}}{67.574}$. *Ans.* 0.23946.

10. $\dfrac{600.45 \times 1.0025}{(37.5)^2 \times \sqrt{617.5}}$.

11. $\sqrt{\dfrac{3.1416 \times 6 \times 27.3}{1.056 \times 2.738 \times 10}}$. *Ans.* 4.2188.

12. $\dfrac{\sqrt[3]{(27.3)^2 \times 23.7}}{\sqrt{(28.92)^3 \times 16.5}}$.

13. The area of a triangle is given by the formula

$$A = \sqrt{s\,(s-a)(s-b)(s-c)},$$

where a, b, c represent the three sides of the triangle and s half their sum. Find A when $a = 617.34$, $b = 345.65$, $c = 467.75$.

Ans. 80127.

14. In Art. 21 it was shown that

$$\tan\frac{A}{2} = \sqrt{\frac{c-b}{c+b}},$$

where b is a side of a right triangle, c the hypotenuse, and A the angle between b and c. Find A when $c = 325.76$, $b = 324.13$.

Ans. $5^\circ\,44'$.

15. Solve the equation $2^x = 3.573$.* *Ans.* 1.8371.

16. Find x in $(3.1416)^x = 9.8697$.

17. Solve the equation

$$3^{2x} - 12 \times 3^x + 11 = 0.$$ *Ans.* $x = 0$, or 2.1827.

18. Solve

$$e^{\frac{x}{2}} + e^{-\frac{x}{2}} = 4, \quad \text{where } e = 2.7183.$$

Ans. $x = \pm\, 2.6339$.

* Such an equation as this, in which the unknown quantity appears as an exponent, is called an *exponential equation.* It is solved by first taking logarithms of both sides of the equation, thus:

$$x \log 2 = \log 3.573,$$

that is, $$0.30103\,x = 0.55303,$$

from which $$x = 1.8371.$$

33. Application of Logarithms. The solution of triangles, which furnishes the most important application of logarithms, will be fully considered later on. At present we give a few miscellaneous problems which are especially adapted to solution by logarithms.

EXERCISE 19

Many of the following list of problems require for their solution the compound interest formula, Problem 10, Exercise 16.

1. Find the amount on $100 for 100 years at 4 % compound interest. ($\log 1.04 = 0.0170333$.) *Ans.* $5050.4.

2. In what time will a sum of money double itself at 8 % compound interest? *Ans.* 9.007 yrs.

3. A man bequeaths $500, which is to accumulate at compound interest until the interest for one year at 5 % will amount to at least $300, after which the yearly interest is to be awarded as a scholarship. How many years must elapse before the scholarship becomes available, assuming that the original bequest is made to earn 5 % compound interest? *Ans.* 51 yrs.

4. At what rate of interest must the bequest in Problem 3 be invested in order that the scholarship may become available in 40 yrs.? *Ans.* 6.4 %.

5. In 1624 the Dutch bought Manhattan Island from the Indians for about $24. Suppose that the Indians had put their money out at compound interest at 7 % and had added the interest to the principal each year, how large would be the accumulated amount in 1910? (From White's Scrap Book of Mathematics.)

Ans. In round numbers $6,000,000,000. The actual valuation of Manhattan and Bronx real and personal property in 1908 was $5,235,399,980.

6. The population of the state of Washington in 1890 was 349,400 and in 1900 it was 518,100. What was the average yearly rate of increase? Assuming the rate of increase to remain the same, what should be the population in 1910? *Ans.* 4 %; 767,970, nearly.

7. The founder of a new faith makes one convert each year, and each new convert makes another convert each year, and so on.

How long would it require to convert the whole earth to the new faith, assuming that the population of the world is 1,500,000,000?

Ans. Between 30 and 31 yrs.

8. The combined wealth of the United States and Europe was estimated (1908) to amount to about \$450,000,000,000. Let us assume that the entire wealth of the world amounts to $\$10^{12}$. How long would it take \$1 put out at compound interest at 3 % to equal or exceed this amount?

Ans. 935 yrs.

34. To Compute a Table of Common Logarithms.

A table of logarithms may be computed from the successive square-roots of 10 and multiplications.

$$10^{\frac{1}{2}} = \sqrt{10} = 3.16228, \quad (1)$$
$$10^{\frac{1}{4}} = \sqrt{10^{\frac{1}{2}}} = \sqrt{3.16228} = 1.77828, \quad (2)$$
$$10^{\frac{1}{8}} = \sqrt{10^{\frac{1}{4}}} = \sqrt{1.77828} = 1.33352, \quad (3)$$
$$10^{\frac{1}{16}} = \sqrt{10^{\frac{1}{8}}} = \sqrt{1.33352} = 1.15478. \quad (4)$$

By definition, the common logarithm of a number is the power to which 10 must be raised to produce that number, hence from

$10^{\frac{1}{16}} = 1.15478,$ $\qquad \log 1.1548^* = \frac{1}{16} = 0.0625;$

$10^{\frac{2}{16}} = 10^{\frac{1}{8}} = 1.33352,$ $\qquad \log 1.3335 = \frac{2}{16} = 0.1250;$

$10^{\frac{3}{16}} = 10^{\frac{2}{16}+\frac{1}{16}} = 10^{\frac{2}{16}} \cdot 10^{\frac{1}{16}} = 1.33352 \times 1.15478 = 1.53992,$

$\log 1.5399 = \frac{3}{16} = 0.1875;$

$10^{\frac{4}{16}} = 10^{\frac{3}{16}+\frac{1}{16}} = 10^{\frac{3}{16}} \cdot 10^{\frac{1}{16}} = 1.53992 \times 1.15478 = 1.77828,$

$\log 1.7783 = \frac{4}{16} = 0.2500.$

Check. $10^{\frac{4}{16}} = 10^{\frac{1}{4}} = 1.77828$ by (2).

$10^{\frac{5}{16}} = 10^{\frac{4}{16}+\frac{1}{16}} = 10^{\frac{4}{16}} \cdot 10^{\frac{1}{16}} = 1.77828 \times 1.15478 = 2.05352,$

$\log 2.0535 = \frac{5}{16} = 0.3125;$

$10^{\frac{6}{16}} = 10^{\frac{5}{16}+\frac{1}{16}} = 10^{\frac{5}{16}} \cdot 10^{\frac{1}{16}} = 2.05352 \times 1.15478 = 2.37136,$

$\log 2.3714 = \frac{6}{16} = 0.3750;$

$10^{\frac{7}{16}} = 10^{\frac{6}{16}+\frac{1}{16}} = 10^{\frac{6}{16}} \cdot 10^{\frac{1}{16}} = 2.37136 \times 1.15478 = 2.73840,$

$\log 2.7384 = \frac{7}{16} = 0.4375;$

$10^{\frac{8}{16}} = 10^{\frac{7}{16}+\frac{1}{16}} = 10^{\frac{7}{16}} \cdot 10^{\frac{1}{16}} = 2.73840 \times 1.15478 = 3.16225,$

$\log 3.1623 = \frac{8}{16} = 0.5000.$

Check. $10^{\frac{8}{16}} = 10^{\frac{1}{2}} = 3.16228$ by (1).

$10^{\frac{9}{16}} = 10^{\frac{8}{16}+\frac{1}{16}} = 10^{\frac{8}{16}} \cdot 10^{\frac{1}{16}} = 3.16228 \times 1.15478 = 3.65174,$

$\log 3.6517 = \frac{9}{16} = 0.5625;$

* Only 5 figures are carried in order to secure accuracy in the last figure of all the numbers in the list.

$$10^{\frac{10}{16}} = 10^{\frac{9}{16}+\frac{1}{16}} = 10^{\frac{9}{16}} \cdot 10^{\frac{1}{16}} = 3.65174 \times 1.15478 = 4.21696,$$
$$\log 4.2170 = \tfrac{10}{16} = 0.6250;$$

$$10^{\frac{11}{16}} = 10^{\frac{10}{16}+\frac{1}{16}} = 10^{\frac{10}{16}} \cdot 10^{\frac{1}{16}} = 4.21696 \times 1.15478 = 4.86966,$$
$$\log 4.8697 = \tfrac{11}{16} = 0.6875;$$

$$10^{\frac{12}{16}} = 10^{\frac{11}{16}+\frac{1}{16}} = 10^{\frac{11}{16}} \cdot 10^{\frac{1}{16}} = 4.86966 \times 1.15478 = 5.62339,$$
$$\log 5.6234 = \tfrac{12}{16} = 0.7500.$$

Check. $10^{\frac{12}{16}} = 10^{\frac{3}{4}} = 10^{\frac{1}{2}} \cdot 10^{\frac{1}{4}} = 3.16228 \times 1.77828 = 5.62342$ by (1) and (2).

$$10^{\frac{13}{16}} = 10^{\frac{12}{16}+\frac{1}{16}} = 10^{\frac{12}{16}} \cdot 10^{\frac{1}{16}} = 5.62342 \times 1.15478 = 6.49382,$$
$$\log 6.4938 = \tfrac{13}{16} = 0.8125;$$

$$10^{\frac{14}{16}} = 10^{\frac{13}{16}+\frac{1}{16}} = 10^{\frac{13}{16}} \cdot 10^{\frac{1}{16}} = 6.49382 \times 1.15478 = 7.49892,$$
$$\log 7.4989 = \tfrac{14}{16} = 0.8750;$$

$$10^{\frac{15}{16}} = 10^{\frac{14}{16}+\frac{1}{16}} = 10^{\frac{14}{16}} \cdot 10^{\frac{1}{16}} = 7.49892 \times 1.15478 = 8.65960,$$
$$\log 8.6596 = \tfrac{15}{16} = 0.9375;$$

$$10^{\frac{16}{16}} = 10^{\frac{15}{16}+\frac{1}{16}} = 10^{\frac{15}{16}} \cdot 10^{\frac{1}{16}} = 8.65960 \times 1.15478 = 9.99993,$$
$$\log 10.000 = \tfrac{16}{16} = 1.0000.$$

Check. $10^{\frac{16}{16}} = 10^1 = 10.$

Each check is on the four lines just preceding, and in each case there is exact agreement to five figures between the results of the check and the result of the line just preceding. The discrepancy in the sixth figure, where it occurs, is due to the fact that all our results are approximations only.

We have thus computed the numbers whose mantissas are distributed between 0 and 1 at intervals of $\frac{1}{16}$.

Similarly, with the aid of

$$10^{\frac{1}{32}} = \sqrt{10^{\frac{1}{16}}} = \sqrt{1.15478} = 1.07468,$$
$$10^{\frac{1}{64}} = \sqrt{10^{\frac{1}{32}}} = \sqrt{1.07486} = 1.03663, \text{ etc.},$$

longer lists of numbers could be computed whose mantissas differ by $\frac{1}{32}$, $\frac{1}{64}$, etc. By continuing this process the interval between successive mantissas can be made small at will. When the difference between successive mantissas has been made sufficiently small, the mantissa corresponding to any number, intermediate to two numbers in the list already found, may be found by interpolation. If

now the numbers on the left are selected at equal intervals and tabulated, together with their mantissas, we shall have a table of common logarithms.*

The characteristics need not be tabulated, for they can always be determined from memory by the two rules of the preceding article. In most of the tables the decimal points are also omitted; for instance, we find in most tables corresponding to

the number	the mantissa
4753	67697,

meaning that .67697 is the fractional part of the logarithm of any number whose significant figures are 4753.

The method just explained, simple as it is, is not the method by which the existing tables of logarithms have actually been calculated. We shall learn later that other methods exist by means of which logarithms may be calculated with much greater ease and speed.

35. Relation between $\log_a N$ and $\log_b N$. We will now show that the logarithms of the same numbers to two different bases are proportional, so that when the logarithm of a number to a given base, as the base 10, is known, the logarithm of the same number to any other base may be obtained from it by multiplying the known logarithm by a certain constant. It is therefore unnecessary to actually construct logarithmic tables to more than one base.

To prove that

$$\log_b N = \mu \log_a N, \text{ where } \mu \text{ is the constant } \frac{1}{\log_a b}.$$

Proof. Let $\log_b N = x$, from which $b^x = N$, (1)

and $\log_a N = y$, from which $a^y = N$. (2)

From (1) and (2)

$$b^x = a^y. \tag{3}$$

Now let

$$b = a^c, \tag{4}$$

* The student will observe that in the original arrangement the numbers on the right (mantissas) are at equal intervals. In this form the table is known as a table of *antilogarithms*. A table of antilogarithms would serve the main purposes of computation as well as a table of logarithms. Tables of antilogarithms have been published and are used by some computers.

and substitute (4) in (3), then

$$(a^c)^x = a^{cx} = a^y,$$

from which $$cx = y,$$

or $$x = \frac{y}{c}, \tag{5}$$

where from (4), $$c = \log_a b.$$

Putting in (5) for x and y their values from (1) and (2), and M for $\frac{1}{c}$ we have

$$\log_b N = \mu \log_a N, \tag{6}$$

where

$$\mu = \frac{1}{\log_a b}.$$

(6) may also be written

$$\log_b N = \frac{\log_a N}{\log_a b}. \tag{7}$$

If in (7) we put $N = a$, we obtain (since $\log_a a = 1$)

$$\log_b a = \frac{1}{\log_a b},$$

that is,

$\log_b a$ and $\log_a b$ are reciprocals.

The constant multiplier μ is called the *modulus* of the system of logarithms whose base is b with reference to the system whose base is a.

36. Natural or Hyperbolic Logarithms. Theoretically any positive number different from 1 may serve as the base of a system of logarithms, but in practice only two systems are used. The first is the *common system*, used exclusively in numerical computations; the other is known as the system of *natural or hyperbolic logarithms*, which is used extensively in theoretical investigations.

The base of the natural system * is an incommensurable number,

* Natural or hyperbolic logarithms are known also as Napierian logarithms in honor of John Napier (1550–1617), though this name is a misnomer, since neither Napier nor any of his contemporaries had any conception of the number e or the system of logarithms which has e for its base. Napier's base is the number 0.367879, which happens to be nearly equal to $\frac{1}{e}$. The discovery of natural logarithms as well as the name is due to Nicolas Mercator (1620–1687).

known as e, the twin sister of the number π, which like π has many remarkable properties. e is defined as the limit which $\left(1+\frac{1}{x}\right)^x$ approaches, as x increases indefinitely. It will be shown later that this limit is equivalent to the series

$$1+1+\frac{1}{2}+\frac{1}{2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{2\cdot 3\cdot 4\cdot 5}+\cdots,$$

from which the approximate value of e is readily found to be

$$e = 2.71828\cdots.$$

By the preceding article we find the modulus of the natural system of logarithms with reference to the common system is

$$\mu_e = \frac{1}{\log 2.71828} = \frac{1}{0.434294} = 2.30259\cdots.$$

Exercise 20

1. Use the results of Art. 34 to compute to four places the number whose common logarithm is $\frac{5}{32} = 0.15625$. *Ans.* 1.4330.

2. By a method similar to that of Art. 34 compute to four places the number whose common logarithm is $\frac{1}{3}$; also to four places the number whose common logarithm is $\frac{1}{9}$. *Ans.* 2.1544, 1.2913.

3. Similarly compute the number whose common logarithm is $\frac{1}{6}$. *Ans.* 1.4678.

4. Given $\log_{10} 2 = 0.30103$; compute $\log_2 10$. *Ans.* 3.3219.

5. Show that $\log_{10} N = \dfrac{\log_e N}{\log_e 10} = 0.434294 \times \log_e N$.

6. Find $\log_e 100$, $\log_e 1000$, $\log_e 0.01$, $\log_e 2$.
Ans. 4.60518, 6.90777, $-$ 4.60518, 0.69315.

7. Use the results of Art. 36 to compute $\log_e 3.65174$.
Ans. 1.29521.

8. Given $\log_{10} 3$; find $\log_9 10$, $\log_{27} 1000$.
Ans. $\dfrac{1}{2\log_{10} 3}$, $\dfrac{1}{\log_{10} 3}$.

9. Show how to obtain:
logarithms to the base 8 from logarithms to the base 2;
logarithms to the base 3 from logarithms to the base 9.
Ans. Divide by 3; multiply by 2.

10. Prove that

$$\log_b a \cdot \log_c b \cdot \log_a c = 1.$$

11. Given $\log 4 = 0.60206$, $\log 9 = 0.95424$; find $\log 6$.

Ans. $\log 6 = 0.77815$.

37. Tables of Logarithmic Trigonometric Functions. In the solution of triangles and elsewhere we constantly encounter expressions involving trigonometric functions. Suppose it were required to compute the value of

$$x = \frac{325.6 \sin 23^\circ 45' \tan 18^\circ 24'}{\cos 37^\circ 30'}.$$

It is plain that we might first find the values of the several trigonometric functions from the table of natural functions; thus, —

$$\sin 23^\circ 45' = 0.4027, \quad \tan 18^\circ 24' = 0.3327, \quad \cos 37^\circ 30' = 0.7934.$$

With these values the expression above written becomes

$$x = \frac{325.6 \times 0.4027 \times 0.3327}{0.7934}.$$

Solving by logarithms we have

$$\begin{aligned}
\log 325.6 \qquad\qquad\qquad\qquad\qquad &= 2.51268 \\
\log 0.4027 = \log \sin 23^\circ 45' \qquad &= 9.60498 - 10 \\
\log 0.3327 = \log \tan 18^\circ 24' \qquad &= 9.52205 - 10 \\
\text{colog } 0.7934 = \text{colog} \cos 37^\circ 30' \qquad &= \underline{0.10051 \qquad\quad} \\
\log x &= 1.74022 \\
x &= 54.983.
\end{aligned}$$

It will be observed that in order to obtain $\log \sin 23^\circ 45'$ we consulted two different tables: first, the table of natural sines, to find $\sin 23^\circ 45' = 0.4027$; second, the table of logarithms, to find $\log 0.4027 = 9.60498 - 10$. To avoid the necessity of referring to two tables, the logarithms of the sines, cosines, tangents and cotangents have been separately calculated and arranged in a new table, known as the table of *logarithmic trigonometric functions* *. The table contains the values of the logarithms of the sines, cosines, tangents and cotangents of angles between 0° and 90°.

We know that the sine and cosine of every angle and the tangent of every angle less than 45°, as well as the cotangent of any angle be-

* The choice of the term is unfortunate. Trigonometric logarithms would be much better. It is of interest to know that Napier's tables, the first tables ever published, were tables of logarithmic trigonometric functions.

tween 45° and 90°, is less than unity, consequently the characteristics of the logarithms of these functions will be negative. To avoid negative characteristics, each negative characteristic has been replaced by a positive characteristic by adding 10. For example, while the true value of

$$\log \sin 23^\circ 45' = -1 + 0.60503 = 9.60503 - 10,$$

the table gives

$$\log \sin 23^\circ 45' = 9.60503.$$

This latter value is called the *tabular logarithmic sine*, hence

To find the true value of a logarithmic sine the corresponding tabular logarithmic sine must be diminished by 10.

Since the secant is the reciprocal of the cosine, and the cosecant the reciprocal of the sine, we have

The logarithmic secant or cosecant may be obtained by taking the cologarithm of the cosine or sine respectively, that is, by subtracting the tabular logarithmic cosine or sine from 10.

38. To Find the Logarithmic Trigonometric Functions of an Angle less than 90°.

(*a*) *When the angle is less than 45°*, we find the number of degrees at the top of the page and the number of minutes in the left-hand column. The values of the log sin, log tan, log cot, log cos, as indicated at the head of the column, are then found in the same line as the minutes.

If seconds are given, they may be reduced to a fractional part of a minute, and the value of the logarithmic function may be found by interpolation, just as was done in finding logarithms of numbers.

Example 1. Find the log sin, log cos, log tan and log cot of $11^\circ 21'$.

Remembering that certain of the characteristics as given in the table are too large by 10, we find from table II

$$\log \sin 11^\circ 21' = 9.29403 - 10,$$
$$\log \tan 11^\circ 21' = 9.30261 - 10,$$
$$\log \cot 11^\circ 21' = 0.69739,$$
$$\log \cos 11^\circ 21' = 9.99142 - 10.$$

EXAMPLE 2. Find log sin $15° 24' 36''$.

From the table we find,

$$\begin{aligned} \log \sin 15° 24' &= 9.42416 \\ \log \sin 15° 25' &= 9.42461 \\ \text{difference for } 1' &= \quad\;\; 45 \end{aligned}$$

$36'' = 0.6'$, difference for $0.6' = 0.6 \times 45 = 27$,

hence $\log \sin 15° 24' 36'' = 9.42416 + 27 = 9.42443$.

EXAMPLE 3. To find log cos $27° 36' 40''$.

From the table we find

$$\begin{aligned} \log \cos 27° 36' &= 9.94753 \\ \log \cos 27° 37' &= 9.94747 \\ \text{difference for } 1' &= \quad\;\; 6 \end{aligned}$$

$40'' = 0.6\frac{2}{3}'$, difference for $0.6\frac{2}{3}' = 4$,

hence $\log \cos 27° 36' 40'' = 9.94753 - 4 = 9.94749$.

Observe that in this example the difference was subtracted from the mantissa of the logarithm of the smaller angle, because the logarithm of the cosine decreases as the angle increases.

(*b*) *When the angle is more than $45°$ and less than $90°$*, we find the degrees at the bottom of the page and the minutes in the right-hand column. The values of the log sine, log tangent, log cotangent and log cosine as indicated at the foot of the column are then found in the same line as the minutes.

EXAMPLE 4. Find log tan $61° 10' 27''$.

From the table we find

$$\begin{aligned} \log \tan 61° 10' &= 0.25923 \\ \log \tan 61° 11' &= 0.25953 \\ \text{difference for } 1' &= \quad\;\; 30 \end{aligned}$$

$27'' = 0.4\frac{1}{2}'$, difference for $0.4\frac{1}{2}' = 4\frac{1}{2} \times 30 = 13.5$,

hence $\log \tan 61° 10' 27'' = 0.25923 + 13.5 = 0.25936$.

The difference 13.5 is additive because the log tangent increases with the angle.

EXAMPLE 5. Find log cot $75° 51' 15''$.

$$\begin{aligned} \log\cot 75° 51' &= 9.40159 \\ \log\cot 75° 52' &= \underline{9.40106} \\ \text{difference for } 1' &= 53 \end{aligned}$$

$15'' = 0.25'$, difference for $0.25' = 13$,

hence $\log\cot 75° 51' 15'' = 9.40146 - 10.$

The difference 13 is subtracted because log cot $75° 52'$ is less than log cot $75° 51'$.

In general when interpolating:

For *sines* and *tangents* the difference must be *added* to the function of the smaller angle, because the sine and tangent of an angle *increases as the angle increases.*

For *cosines* and *cotangents* the difference must be *subtracted* from the function of the smaller angle, because the cosine and cotangent of the angle *decreases as the angle increases.*

39. To Find the Angle Corresponding to a Given Logarithmic Trigonometric Function.

When the given number can be found in the table, the number of degrees is found at the top or bottom of the page, according as the name of the function appears at the top or bottom of the column. The number of minutes is found in the same line with the given function, — in the left column if the degrees are taken from the top of the page, in the right column if from the bottom.

When the given number cannot be found in the table, two consecutive numbers can always be found, one of which is slightly smaller, the other slightly larger than the given number. The required angle can then be found by interpolation.

EXAMPLE 1. Given $\log\sin x = 8.73997$; to find x.

The number 8.73997 is found in the first column on page 30 of the tables. This column has log sin written at the top and log cos at the bottom. Since the given number is to be a log sine, the degrees are taken from the top of the page and the number of minutes from the left-hand column in line with the number 8.73997. We thus find $x = 3° 09'$.

EXAMPLE 2. Given $\log\cos y = 8.73997$; to find y.

The given number is the same as in Example 1, but this time it represents a log cosine. Therefore we take for the required angle

the number of degrees at the bottom of the page and the number of minutes found in the right-hand column, opposite the number in the table. Result, $y = 86° \, 51'$.

It should be observed that the given mantissa 73997 appears a second time in the table, namely, on page 60, first column. But the numbers in that column have the characteristic 9, while the given characteristic is 8. Beginners sometimes overlook the characteristic and are thus led into mistakes.

EXAMPLE 3. Given log tan $x = 0.08685$; to find x.

The number 0.08685 cannot be found in the table, but on page 66, third column, are found the next smaller number 0.08673 and the next larger number 0.08699. The difference between these two is 26, and to the smaller of the two corresponds the angle $50° \, 41'$, that is,

$$\begin{array}{lrcr} & \text{mantissa log tan } 50° \, 41' & = & 08673 \\ \text{but} & \text{mantissa log tan } x & = & \underline{08685} \\ & \text{difference} & = & 12. \end{array}$$

If we denote by d'' the difference between $50° \, 41'$ and x, the principle of proportional parts gives us

$$d'' : 12 = 60'' : 26, \text{ that is, } d'' = \tfrac{12}{26} \times 60'' = 28'',$$

and the required angle is $x = 50° \, 41' \, 28''$.

EXAMPLE 4. Given log cos $x = 9.87561 - 10$; to find x.

$$\begin{array}{lrcr} & \text{mantissa log cos } 41° \, 19' & = & 87568 \\ & \text{mantissa log cos } 41° \, 20' & = & \underline{87557} \\ & \text{difference for } 1' \text{ or } 60'' & = & 11. \\ \text{Also} & \text{mantissa log cos } 41° \, 19' & = & 87568 \\ & \text{mantissa log cos } x & = & \underline{87561} \\ & \text{difference for } d'' & = & 7 \end{array}$$

and $d'' : 7 = 60'' : 11, \qquad d'' = \tfrac{7}{11} \times 60'' = 38''$,

hence the required angle is $x = 41° \, 19' \, 38''$.

EXERCISE 21

From the table of logarithmic trigonometric functions find:

1. log sin $13° \, 24'$, log cos $25° \, 12'$, log tan $10° \, 02'$, log cot $17° \, 00'$.
   Ans. 9.36502, 9.95657, 9.24779, 0.51466.
2. log sin $72° \, 11'$, log tan $46° \, 17'$, log cot $65° \, 13'$, log cos $67° \, 59'$.
   Ans. 9.97866, 0.01946, 9.66437, 9.57389.

3. log sin $25° 12' 30''$, log tan $36° 30' 15''$, log sin $70° 15' 40''$.
Ans. 9.62932, 9.86928, 9.97370.

4. log cos $16° 26' 45''$, log cot $9° 27' 42''$, log cot $54° 25' 09''$.
Ans. 9.98186, 0.77818, 9.85456.

5. log sin $42° 15' 10''$, log cos $17° 10' 54''$, log tan $51° 18' 57''$.
Ans. 9.82763, 9.98017, 0.09653.

By means of the table find x, when there is given:

6. $\log \sin x = 9.77980$, $\log \tan x = 0.40017$, $\log \cos x = 9.79558$.
Ans. $37° 02'$, $68° 18'$, $51° 21'$.

7. $\log \sin x = 9.68931$, $\log \cos x = 9.89609$, $\log \tan x = 0.16999$.
Ans. $29° 16' 30''$, $38° 04' 30''$, $55° 56' 15''$.

8. $\log \tan x = 0.30562$, $\log \sin x = 8.97296$, $\log \sin x = 9.97296$.
Ans. $63° 40' 36''$, $5° 23' 30''$, $69° 59' 24''$.

9. $\log \sin x = \log 2 + \log \sin 32° 10' 15'' + \log \cos 32° 10' 15''$.
Ans. $x = 64° 20' 30''$.

10. Verify the relation

$$\log \cos 27° 10' 40'' = \log (\cos 13° 35' 20'' - \sin 13° 35' 20'') + \log (\cos 13° 35' 20'' + \sin 13° 35' 20'').$$

(Suggestion. Use the natural function table to find the quantities within the parentheses.)

11. Given $\tan x = 4.6525$; find x without using a table of natural functions.

12. Given $(\sin x)^{\frac{2}{3}} = 0.253$; find x. *Ans.* $x = 7° 18' 40''$.

40. Logarithmic Functions of Angles near 0° or 90°. When the angle is very small, say less than 2°, the logarithms of the sine, tangent and cotangent vary so rapidly that the principle of proportional parts fails to apply with sufficient accuracy, causing the results obtained by interpolation to be unreliable. The same remark applies to the logarithms of the cosine, cotangent and tangent of an angle near 90°, say between 88° and 90°. In such cases tables may be used which give the required functions for sufficiently small intervals of the angle, say for each second. When such tables are not available the following rules may be employed.

If the angle is less than 2° and only tenths of minutes are considered, —

The log sin or log tan of the angle may be found by increasing the logarithm of the number of minutes in the angle by 6.46373–10.

Conversely,

The angle corresponding to a given log sin or log tan may be found by diminishing the given log sin or log tan by 6.46373–10. *The resulting number is the logarithm of the number of minutes in the required angle.*

EXAMPLE 1. To find log sin 0° 50.7′.

$$\begin{aligned} \log 50.7 &= 1.70501 \\ &\underline{\ 6.46373 - 10} \\ \log \sin 50.7' &= 8.16874 - 10 \end{aligned}$$

EXAMPLE 2. Given log tan x = 8.50724 − 10; to find x.

$$\begin{aligned} \log \tan x &= 8.50724 - 10 \\ &\underline{\ 6.46373 - 10} \\ \log x &= 2.04351 \\ x &= 110.5' = 1^\circ\ 50.5'. \end{aligned}$$

These rules are based on the theorem, which will be proved later, that the sine and tangent of a small angle are each approximately equal to the length of the arc subtended by the angle at the center of a circle whose radius is the unit of measure, so that for very small angles the measure of the arc may be substituted for either the sine or the tangent of the corresponding angle. Now the measure of an arc in terms of the radius is found by multiplying the number of minutes in the arc by 0.0002909, for since the semi-circumference (the arc subtending an angle of 180° = 10800′) measures 3.14159 . . . when the radius is 1, each minute of arc measures 3.14159 ÷ 10800 = 0.0002909. Log 0.0002909 = 6.46373 − 10, consequently if the log sin or log tan of a small angle is increased by this amount the result will be the logarithm of the number of minutes in the corresponding angle.

When the angle is less than 2° and the exact number of seconds are considered the foregoing rule fails. In such cases a special table, known as the S and T table, is commonly employed.

41. Use of the S and T Table (Table III). For small angles the ratio of either the sine or tangent to the length of the arc of unit radius is nearly constant, that is

$$\frac{\sin x}{x}, \text{ as well as } \frac{\tan x}{x}, \text{ is nearly constant,}$$

x being the ratio of the length of the arc to the radius.

On taking logarithms we have
log sin x − log x, as well as log tan x − log x, is nearly constant. Let us put

$$\log \sin x - \log x = S, \text{ and } \log \tan x - \log x = T,$$

then on transposing log x we obtain

$$\log \sin x = \log x + S \tag{1}$$

and

$$\log \tan x = \log x + T. \tag{2}$$

The values of S and T, corresponding to various values of x expressed in seconds, have been carefully calculated and assembled in Table III. The given angle x is first reduced to seconds, and the corresponding value of S or T is then taken from the table. This value increased by log x gives log sin x or log tan x, as the case may be.

Conversely, if log sin x or log tan x is given and x is to be found, we take S or T from the table, subtract it from log sin x or log tan x, as the case may be, and obtain log x, from which x is found.

The log cosine or log cotangent of an angle near 90° may be found from the same table by substituting for the cosine of the angle the sine of its complement, and for the cotangent of the angle the tangent of its complement. All this will be better understood by following through the examples which are worked out below.

The characteristics of S and T are negative, so that −10 must be appended to each value taken from the table.

Example 1

Given $x = 0° \; 56' \; 26''$; to find log sin x.

$$x = 0° \; 56' \; 26'' = 3386''.$$

Applying (1),	$\log x = 3.52969$
From Table III,	$S = 4.68556 - 10$
	$\log \sin x = 8.21525 - 10.$

Example 2

Given $\log \sin x = 8.21524 - 10$; to find x.

	$\log \sin x = 8.21524 - 10$
From Table III,	$S = 4.68555 - 10$
By (1),	$\log x = 3.52969$

$$x = 3386'' = 0° \; 56' \; 26''.$$

EXAMPLE 3

To find log tan $1^\circ\ 10'\ 51''$.

$$1^\circ\ 10'\ 51'' = 4251''.$$

$$\begin{array}{lr} \text{Applying (2),} & \log 4251 = 3.62849 \\ \text{From Table III,} & T = 4.68564 - 10 \\ \hline & \log \tan 1^\circ\ 10'\ 51'' = 8.31413 - 10. \end{array}$$

EXAMPLE 4

Given $\log \tan x = 8.31413 - 10$; to find x.

$$\begin{array}{lr} & \log \tan x = 8.31413 - 10 \\ \text{From Table III,} & T = 4.68564 - 10 \\ \hline \text{Subtracting,} & \log x = 3.62849 \end{array}$$

$$x = 4251'' = 1^\circ\ 10'\ 51''.$$

EXAMPLE 5

To find log cos $89^\circ\ 25'\ 11''$.

$$\log \cos 89^\circ\ 25'\ 11'' = \log \sin 0^\circ\ 34'\ 49''$$

$$0^\circ\ 34'\ 49'' = 2089''.$$

$$\begin{array}{lr} \text{Applying (1),} & \log 2089 = 3.31994 \\ \text{From Table III,} & S = 4.68557 - 10 \\ \hline & \log \sin 0^\circ\ 34'\ 49'' = 8.00551 - 10 \\ & \log \cos 89^\circ\ 25'\ 11'' = 8.00551 - 10. \end{array}$$

EXAMPLE 6

$\log \cos x = 8.00551 - 10$; to find x.

$$\log \cos x = \log \sin (90^\circ - x).$$

$$\begin{array}{lr} & \log \sin (90^\circ - x) = 8.00551 - 10 \\ \text{From Table III,} & S = 4.68557 - 10 \\ \hline & \log (90^\circ - x) = 3.31994 \\ & 90^\circ - x = 2089'' = 0^\circ\ 34'\ 49'' \\ & x = 90^\circ - 0^\circ\ 34'\ 49'' = 89^\circ\ 25'\ 11''. \end{array}$$

EXAMPLE 7

To find log cot $1^\circ\ 0'\ 29''$.

$$\log \cot 1^\circ\ 0'\ 29'' = \log \frac{1}{\tan 1^\circ\ 0'\ 29''}$$

$$= -\log \tan 1^\circ\ 0'\ 29''.$$

Then, as in Example 3,

$$\log \tan 1^\circ\ 0'\ 29'' = 8.24541 - 10$$
$$\log \cot 1^\circ\ 0'\ 29'' = 10 - 8.24541$$
$$= 1.75459.$$

Example 8

$\log \cot x = 1.75459$; to find x.

$$\log \cot x = -\log \tan x = 1.75459,$$
$$\log \tan x = 8.24541 - 10.$$

Then, as in Example 4,

$$x = 1^\circ\ 0'\ 29''.$$

Exercise 22

1. Find $\log \sin 1^\circ\ 20'\ 02''$, $\log \tan 0^\circ\ 45'\ 45''$, $\log \cos 88^\circ\ 54'\ 36''$.
Ans. $8.36696 - 10$, $8.12414 - 10$, $8.27928 - 10$.

2. Find $\log \sin 0^\circ\ 50'\ 50''$, $\log \cos 89^\circ\ 01'\ 55''$, $\log \tan 1^\circ\ 15'\ 17''$.

3. What would have been the error in each of the functions in 2, if their values had been found from table II by interpolation?

4. Find each of $\log \tan 88^\circ\ 05'\ 20''$, $\log \cot 89^\circ\ 16'\ 50''$ from Table III, and check your result by Table II.

5. Given $\log \sin x = 8.22925 - 10$, $\log \tan y = 8.43340 - 10$; find x and y. *Ans.* $x = 0^\circ\ 58'\ 17''$, $y = 1^\circ\ 33'\ 14''$.

42. Historical Note. With the awakening of science in the sixteenth century, measurement became more precise, the resulting numbers more complex, and computation more and more tedious and time-consuming. The demand for shorter methods of computation than were then known led to the invention of loga ithms. It is therefore not very strange that the method of logarithms should have been developed independently and almost simultaneously by two mathematicians, John Napier, a Scotchman, and Jost Buergi, a German. Napier's tables were published in 1614. Buergi's tables were computed before 1611 but not published till 1620. Both Napier's and Buergi's tables were soon superseded by Briggs' tables. Briggs' tables contained fourteen place logarithms of all numbers from 1 to 20,000 and from 90,000 to 100,000. Briggs' tables were completed by Adrian Vlacq (1628), who shortened Briggs' tables to ten places,

and computed the logarithms of the remaining numbers from 20,000 to 90,000.

Briggs' and Vlacq's tables are substantially the same as the tables in use to-day, though the tables have been checked and parts of them recomputed many times. The most complete check was undertaken by the French authorities in 1784. It required the work of nearly one hundred mathematicians and computers for over two years. The resulting tables, giving fourteen place logarithms of all integers from 1 to 200,000, besides natural sines and logarithmic sines and tangents, have never been published. Two manuscript copies are preserved, one at the Observatory, the other at the Institute in Paris.

CHAPTER V

LOGARITHMIC SOLUTION OF RIGHT TRIANGLES AND APPLICATIONS

43. Logarithmic Solution of Right Triangles. In Article 20 it was shown how to solve right triangles by means of natural functions. Now we shall employ logarithms, which as a rule shortens the work. We shall illustrate each of the cases which may arise by an example.

EXAMPLE 1.

Given $a = 316.5$,
$c = 521.2$.

Required $A = 37° \ 23.5'$,
$B = 52° \ 36.5'$,
$b = 414.1$.

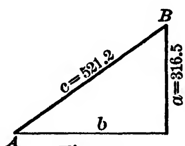


Fig. 30.

Solution.

(*a*) To find A and B.

$$\sin A = \frac{a}{c}.$$

$$\begin{aligned} \log \sin A &= \log a + \operatorname{colog} c. \\ \log a &= 2.50037 \\ \operatorname{colog} c &= 7.28300 - 10 \\ \hline \log \sin A &= 9.78337 - 10 \\ A &= 37° \ 23.5'. \\ B &= 52° \ 36.5'. \end{aligned}$$

(*b*) To find b.

$$\cos A = \frac{b}{c}, \text{ or } b = c \cos A.$$

$$\begin{aligned} \log b &= \log c + \log \cos A. \\ \log c &= 2.71700 \\ \log \cos A &= 9.90009 - 10 \\ \hline \log b &= 2.61709 \\ b &= 414.1. \end{aligned}$$

(*c*) Check. If our result for b is correct, it must satisfy the relation $a^2 + b^2 = c^2$, or $c^2 - b^2 = (c + b)(c - b) = a^2$,

from which $$\log (c + b) + \log (c - b) = 2 \log a.$$

$$\begin{aligned} c + b &= 521.2 + 414.1 = 935.3, \\ c - b &= 521.2 - 414.1 = 107.1. \end{aligned}$$

$$\begin{aligned} \log (c + b) &= 2.97095 \\ \log (c - b) &= 2.02979 \\ \hline & 5.00074 \end{aligned} \qquad \begin{aligned} \log a &= 2.50037 \\ & \quad 2 \\ \hline & 5.00074 \end{aligned}$$

Since b checks, A may be assumed to be correct also, for b was found from A.

EXAMPLE 2.

Given $a = 6.325$,
$b = 7.328$.

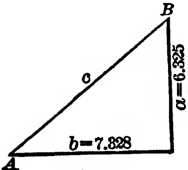


Fig. 31.

Required $A = 40° \, 47.9'$,
$B = 49° \, 12.1'$,
$c = 9.680$.

Solution.

(a) To find A and B.

$$\tan A = \frac{a}{b}.$$

$$\begin{aligned}
\log \tan A &= \log a + \operatorname{colog} b. \\
\log a &= 0.80106 \\
\operatorname{colog} b &= \underline{9.13501 - 10} \\
\log \tan A &= 9.93607 - 10 \\
A &= 40° \, 47.9'. \\
B &= 49° \, 12.1'.
\end{aligned}$$

(b) To find c.

$$\sin A = \frac{a}{c}, \text{ or } c = \frac{a}{\sin A}.$$

$$\begin{aligned}
\log c &= \log a + \operatorname{colog} \sin A. \\
\log a &= 0.80106 \\
\operatorname{colog} \sin A &= \underline{0.18483} \\
\log c &= 0.98589 \\
c &= 9.680.
\end{aligned}$$

(c) Check. $c^2 - b^2 = a^2$, $\log(c + b) + \log(c - b) = 2 \log a$.

$$\begin{aligned}
c + b &= 9.680 + 7.328 = 17.008, \\
c - b &= 9.680 - 7.328 = 2.352.
\end{aligned}$$

$$\begin{aligned}
\log(c + b) &= 1.23065 \\
\log(c - b) &= \underline{0.37144} \\
& 1.60209
\end{aligned}
\qquad
\begin{aligned}
\log a &= 0.80106 \\
& \underline{2} \\
& 1.60212
\end{aligned}$$

Here there is a slight discrepancy in the check. The discrepancy is in the last figure only; that is, if the mantissas of the final results are cut down to four figures, each side is 1.6021. Agreement in the first four figures of the mantissas of the final logarithms of the check is all that can be expected when a five-place table is used.

EXAMPLE 3.

Given $c = 35.145$,
$A = 25° \, 24' \, 30''$.

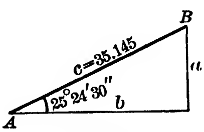


Fig. 32.

Required
$a = 15.079$,
$b = 31.745$,
$B = 64° \, 35' \, 30''$.

Solution.

(a) To find a.

$$\sin A = \frac{a}{c}.$$

$$\begin{aligned}
\log a &= \log c + \log \sin A \\
\log c &= 1.54586 \\
\log \sin A &= \underline{9.63252 - 10} \\
\log a &= 1.17838 \\
a &= 15.079.
\end{aligned}$$

(b) To find b.

$$\cos A = \frac{b}{c}.$$

$$\begin{aligned}
\log b &= \log c + \log \cos A. \\
\log c &= 1.54586 \\
\log \cos A &= \underline{9.95582 - 10} \\
\log b &= 1.50168 \\
b &= 31.745.
\end{aligned}$$

(*c*) Check.

$$\log (c + b) + \log (c - b) = 2 \log a.$$

$$c + b = 66.890, \quad c - b = 3.400.$$

$$\log (c + b) = 1.82536 \qquad \log a = 1.17838$$

$$\log (c - b) = \underline{0.53148} \qquad \underline{2}$$

$$2.35684 \qquad 2.35676$$

EXAMPLE 4.

Given $b = 25.01$,
$B = 65° \, 10'$.

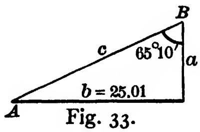


Fig. 33.

Required $a = 11.57$,
$c = 27.56$,
$A = 24° \, 50'$.

Solution.

(*a*) To find *a*.

$$\tan B = \frac{b}{a}.$$

$$\log a = \log b + \log \cot B.$$

$$\log b = 1.39811$$

$$\log \cot B = \underline{9.66537 - 10}$$

$$\log a = 1.06348$$

$$a = 11.574.$$

(*b*) To find *c*.

$$\sin B = \frac{b}{c}.$$

$$\log c = \log b + \operatorname{colog} \sin B.$$

$$\log b = 1.39811$$

$$\operatorname{colog} \sin B = \underline{0.04214}$$

$$\log c = 1.44025$$

$$c = 27.56.$$

(*c*) Check.

$$\log (c + b) + \log (c - b) = 2 \log a.$$

$$c + b = 52.568, \quad c - b = 2.548.$$

$$\log (c + b) = 1.72072 \qquad \log a = 1.06348$$

$$\log (c - b) = \underline{0.40620} \qquad \underline{2}$$

$$2.12692 \qquad 2.12696$$

EXAMPLE 5.

Given $c = 34.57$,
$b = 34.04$.

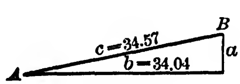


Fig. 34.

Required $A = 10° \, 02.7'$
$B = 79° \, 57.3'$
$a = 6.030$.

Solution. In this problem the given side and hypotenuse are so nearly equal that the method employed in Example 3 does not give sufficiently accurate results. We therefore use the formulas of Art. 21.

(*a*) To find A and B.

$$\tan\frac{A}{2} = \sqrt{\frac{c-b}{c+b}}.$$

$$\log\tan\frac{A}{2} = \frac{1}{2}[\log(c-b) + \text{colog}(c+b)].$$

$$c - b = 34.57 - 34.04 = 0.53,$$
$$c + b = 34.57 + 34.04 = 68.61.$$

$$\log(c-b) = 9.72428 - 10$$
$$\text{colog}(c+b) = 8.16361 - 10$$
$$2)\underline{17.88789 - 20}$$
$$\log\tan\tfrac{1}{2}A = 8.94394 - 10$$

$\frac{1}{2}A = 5°\ 01'\ 22''$

$A = 10°\ 02'\ 44''$, or $10°\ 2.7'$ to the nearest tenth minute.

$B = 79°\ 57'\ 16''$, or $79°\ 57.3'$ to the nearest tenth minute.

(*b*) To find *a*.

$$\sin A = \frac{a}{c}.$$

$$\log a = \log c + \log\sin A.$$
$$\log c = 1.53870$$
$$\log\sin A = 9.24163 - 10$$
$$\log a = 0.78033$$
$$a = 6.030.$$

(*c*) Check.

$$\log(c+b) + \log(c-b) = 2\log a.$$

$$c + b = 34.57 + 34.04 = 68.61,$$
$$c - b = 34.57 - 34.04 = 0.53.$$

$$\log(c+b) = 1.83639$$
$$\log(c-b) = 9.72428 - 10$$
$$1.56067$$

$$\log a = 0.78033$$
$$2$$
$$1.56066$$

Exercise 23

Solve the following problems by logarithms, computing all angles to the nearest second and all sides to five significant figures. It is expected that the student check his results when no answers are given.

1. $a = 168.92$, $c = 289.64$.
   Find $A = 35°\ 40'\ 33''$, $B = 54°\ 19'\ 27''$, $b = 235.28$.
2. $a = 43.148$, $b = 84.107$. Find $A = 27°\ 09'\ 29''$, $c = 94.530$.
3. $b = 2.5346$, $c = 3.7132$. Find a, A, B.
4. $a = 0.37640$, $b = 0.28634$. Find A and B.
5. $a = 547.5$, $A = 32°\ 15'\ 24''$. Find $b = 867.5$, $c = 1025.8$.
6. $a = 6700$, $B = 27°\ 30'$. Find b and c.

7. $c = 672.34$, $A = 35° 16' 25''$.
Find $B = 54° 43' 35''$, $a = 388.26$, $b = 548.90$.

8. $c = 1.001$, $B = 45° 45' 45''$. Find a and b.

9. $c = 369.27$, $b = 235.64$. Find $A = 50° 20' 52''$, $a = 284.31$.

10. $c = 5464.35$, $a = 5452.13$. Find $B = 3° 49' 57''$.

44. Number of Significant Figures. Numerical problems are of two kinds:

(*a*) Those in which the given numbers are *exact* numbers.

(*b*) Those in which the given numbers are *approximate* only. For example, when we say that each of the sides of a hexagon inscribed in a circle with unit radius is 1, and each angle 120°, 1 and 120 are exact numbers, that is, the sides in question are to be considered neither more nor less than 1, and the angles neither more nor less than 120°. On the other hand, when we say that the side of a field measures 631.7 feet and the angle at a corner 73° 37′, the numbers 631.7 and 73° 37′ are mere approximations. So far as we know the exact length of the measured side may be any number between 631.65 and 631.75 and the measured angle may have any value between 73° 36.5′ and 73° 37.5′.

I. *When the given numbers of a problem are exact numbers*, the results asked for can be carried out to as many significant figures as the number of figures in the mantissas of the logarithms used in the solution. In this book, where the computations are based on a five-place table,* lengths must be limited to five significant figures and angles to seconds. Even then the fifth figure and the number of seconds cannot always be relied upon.

II. *When the given numbers of a problem are the results of measurement*, the answers need not contain more significant figures than the least accurate of the given parts. Thus, if one side of a triangle is measured to the nearest inch and another to the nearest tenth of an inch, the answer for the third side need only be given to the nearest inch. The following directions will assist the student to make consistent measurements and to avoid useless calculations.

1. Distances expressed to *three figures* call for angles expressed to the nearest *five minutes*, and vice versa.

* Five-place tables answer most of the demands of applied science. The instruments ordinarily used by engineers read angles to the nearest minute only.

2. Distances expressed to *four figures* call for angles expressed to the nearest *tenth of a minute*,* and vice versa.

3. Distances expressed to *five figures* call for angles expressed to the nearest *second*, and vice versa.

4. Distances expressed to *six figures* call for angles expressed to the nearest *tenth of a second*, and vice versa. A six-place table must be used to obtain like accuracy in the answers.

5. Distances expressed to *seven figures* call for angles expressed to the nearest *hundredth of a second*, and vice versa. A seven-place table is necessary to obtain like accuracy in the answers.

In this connection the student should observe that, whenever a number is the result of measurement or other approximation, a cipher to the right of a decimal fraction has a distinct significance and cannot be dropped at will, as is customary in dealing with exact numbers. For example, the square root of 3 is approximately represented by each of the numbers 1.7, 1.73, 1.732, 1.7320, 1.73205, etc., the approximation being closer the more figures we write; but 1.7, when used as an approximation for $\sqrt{3}$, has not the same meaning as 1.70, for the former means that $\sqrt{3}$ has some value between 1.65 and 1.75, while the latter means that the number represented has some value between 1.695 and 1.705, which is not true. Similarly the numbers 62, 62.0, 62.00, when they represent measures of distances or other quantities, are not equivalent. The first implies that the measurement has been carried out to the nearest unit, the second to the nearest tenth, and the third, 62.00, that the measurement has been made to the nearest hundredth of a unit.

45. Applied Problems Involving Right Triangles. The following six sections deal with applied problems involving the solution of right triangles. The problems are grouped with reference to the questions dealt with, and the problems in each set are so arranged that the more difficult come last. It is not expected that any one student work every problem, but only as many as may be necessary to make him reasonably familiar with the method of solving right triangles by means of logarithms. After that an additional hour or two may profitably be spent in the analysis of the more difficult problems involving two or more right triangles. In problems where no answer is given the *result must be checked* by the student.

* The nearest 10″ is somewhat closer.

46. Heights and Distances

EXERCISE 24

1. From a point 185 feet from the foot of a wireless telegraph mast, the top of the mast was found to form an angle of 52°. Find the height of the mast. *Ans.* 237 ft.

2. A man walking along a straight road observes a church in a direction making an angle of 50° with the road. After walking another mile, he comes to the crossroad on which the church is located. The roads cross at right angles. How far is the church from the intersection of the roads? *Ans.* 1.19 miles.

3. The summit of a mountain, known to be 14,450 feet high, is seen at an angle of elevation of 29° 15′ from a camp located at an altitude of 6935 feet. Compute the air-line distance from the camp to the summit of the mountain. *Ans.* 2.9 miles.

4. The ratio of the height of a roof to its span is one-fourth (quarter pitch), what is the inclination of the roof to the horizontal line? *Ans.* 26° 34′.

5. During a storm a tree was broken into two parts which remained connected. The broken part made an angle of 35° with the ground and its top reached a mark 165 feet from the foot of the tree. Required the height of the remaining stump and the height of the tree before it broke. *Ans.* 116 ft., 317 ft.

6. In constructing a grand-stand, timbers 28 ft. long are to be inclined at an angle of 25° and supported by four uprights, one at each end and two at equal distances between the two ends. How far apart must the uprights be placed and what are their lengths, the shortest being 6 ft. long?

7. A flagpole 25 ft. long stands on a building whose height is unknown. From a point at the same level as the foot of the building the angles of elevation of the top and bottom of the flagpole are measured and are found to be 57° and 53° respectively. Required the height of the building.

Ans. By natural functions, 156 ft.

8. From the top of a tower the angle of depression of a point in the same horizontal plane with the base of the tower is observed to be 47° 13′. What will be the angle of depression of the same point as seen from a position halfway up the tower? *Ans.* 28° 23′.

9. A spherical balloon whose radius is 10 ft. subtends an angle of $1^\circ 46'$, while from the same position and at the same time the angle of elevation of the center of the balloon is 54°. Determine the height of the center of the balloon. *Ans.* 525 ft.

10. An observer finds that the top of a spire due south of him has an angle of elevation of $25^\circ 36'$. He goes to a point 650 ft. east of his first position and now finds that the spire bears $40^\circ 12'$ southwest. Find the height of the spire.

11. It was found that the shadow of a tall factory chimney lengthened 85 ft. while the sun's elevation changed from 59° to 42°. Required the height of the chimney. *Ans.* 167 ft.

47. Problems for Engineers. It is suggested that the student use the graphic method in checking the problems in this set to which no answers are given.

EXERCISE 25

1. A branch railroad is to be constructed from a point A to a second point B which is 5.95 miles east and 9.36 miles north of the first. What will be the direction of the road, assuming that it follows a straight line? *Ans.* N. $32^\circ 27'$ E.

2. To determine the width of a stream a surveyor measures a line AB 375 ft. long along one bank. At B he turns a right angle and his assistant places a stake in the line of sight at C on the opposite bank of the stream. The angle BAC measures $64^\circ 42'$. How wide is the stream? *Ans.* 793 ft.

3. On a map on which 1 inch represents 1000 ft., contour lines are drawn for differences of 100 ft. in altitude. What is the actual inclination of the surface represented by that portion of the map at which the contour lines are one-fourth inch apart?

Ans. $21^\circ 48'$.

4. A bolt 2 inches in diameter has six threads to the inch. What is the inclination of the thread to a cross section of the bolt?

Ans. $1^\circ 31.2'$.

5. A car track runs from A to B, a horizontal distance of 1275 ft. at an incline of $7^\circ 45'$, and then from B to C a horizontal distance of 1585 ft. C is known to be 509 ft. above A. What is the average inclination of the track from B to C?

6. Two towns A and B, of which B is 25 miles northeast of A, are to be connected by a new road. Ten miles of the road is constructed from A in the direction N. 23° E., what must be the length and direction of the remainder of the road, assuming that it follows a straight line? *Ans.* 16.17 miles, N. 58° 23.8′ E.

7. A surveyor wishes to ascertain the distance between two inaccessible objects A and B. He starts from a point C in a straight line with A and B and measures in a direction at right angles to AB a distance CD equal to 500 ft. At D he measures the angles subtended by AC and BC and finds them to be 75° 35′ and 34° 46′ respectively. Find the distance between A and B on the supposition,—

(*a*) That C is between A and B,

(*b*) That B is between A and C.

Ans. (*a*) 2292, (*b*) 1598.

8. One end of a connecting rod AB, 5 ft. long, is fastened to a crank BO, 1 ft. long, while the other end is fastened to a crosshead A which is constrained to move along AO. How far from the extreme position P of the crosshead will A be,—

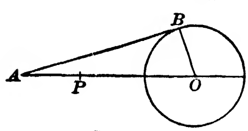


Fig. 35.

(*a*) When OB is perpendicular to AB?

(*b*) When angle $BOA = 60°$?

Ans. (*a*) $\sqrt{26} - 4 = 1.099$ ft.

(*b*) $\frac{1}{2}\sqrt{97} + \frac{1}{2} - 4 = 1.424$ ft.

(Suggestion. In case (*b*) drop a perpendicular BC from B to AO, find OC and BC, then from the triangle ABC find AC.)

9. Two railroad tracks intersect at an angle of 54° 16′. They are to be connected by a curve AB of 100 ft. radius. Find how far from the intersection point O of the tracks the curve begins and the length of the curve.

Ans. $OB = 51.25$ ft. Arc $AB = 94.71$ ft.

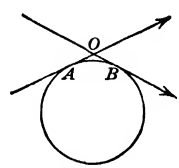


Fig. 36.

10. Two pulleys whose radii are 18 inches and 30 inches respectively are 8 ft. apart from center to center. Find the length of the belt connecting the pulleys,—

(*a*) If the pulleys are to turn in the same direction,

(*b*) If the pulleys are to turn in opposite directions.

48. Applications from Physics.

EXERCISE 26

1. The horizontal distance between the two extreme positions of a pendulum 39.1 inches long is 5.73 inches. Through what angle does it swing? *Ans.* $8° \ 24'$.

2. Two forces of 10 and 24 lbs. respectively act at right angles to each other. Find the resultant force, and also the angle which the resultant makes with the first of the two given forces.

Ans. 26 lbs., $67° \ 23'$.

3. What force is necessary to roll a barrel weighing 500 lbs. onto a platform 6 ft. high along an inclined ladder 12 ft. long?

Ans. 250 lbs.

4. A ball weighing 300 lbs. rests on a smooth plane inclined at an angle of $12° \ 30'$ to the horizontal. What force is necessary to keep the ball from rolling down the plane,

(*a*) If the force acts parallel to the inclined plane?

(*b*) If the force acts in a horizontal direction?

Ans. (*a*) 64.93 lbs., (*b*) 66.51 lbs.

5. A block of wood rests on an adjustable inclined plane. As the inclination of the plane reaches $29° \ 37'$ the block begins to slide. Find the coefficient of friction.*

6. An automobile moving at the rate of 45 miles per hour is overtaken by a shower. As seen from the automobile the raindrops seem to come down at an angle of $30°$ with the vertical. Find the velocity of the raindrops, assuming that their actual direction is vertical. *Ans.* 38.1 ft. per sec.

7. Through what angle must a fir log 30 ft. long and 54 inches in diameter, standing on end, be tilted before it begins to fall? The log is assumed to be cylindrical in shape. *Ans.* $8° \ 37' \ 40''$.

8. According to Wollaston the intensity of sunlight is equal to 61,000 standard candles acting at a distance of 1 meter. What is the intensity of sunlight striking a surface at an angle of $31° \ 08' \ 27''$?

Ans. 52210 c.p.

9. The fans of a windmill are inclined $25°$ to the plane of the wheel which is at right angles to the direction from which the wind

* The coefficient of friction is equal to the tangent of the angle of inclination of the plane on which the block rests.

blows. What fraction of the wind's force is effective in turning the wheel? *Ans.* 0.383.

10. A weight of 437 lbs. is suspended and pushed $17° 30'$ out of the vertical by a horizontal force. Required the horizontal force necessary to hold the body in this position.

11. What is the displacement CM of a ray of light AB in passing through a glass plate PQ, 0.215 inches thick, at an angle of $55° 47'$ with the perpendicular EB, the index of refraction * from air to glass being approximately $\frac{3}{2}$.

Ans. 0.098 in.

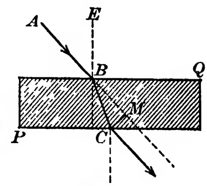


Fig. 37.

12. A sail ship sails against the wind at an angle of 60°. The sails are set so as to make an angle of 15° with the direction OV of the ship. What part of the wind's force is effective in producing the forward motion of the ship? *Ans.* 0.183.

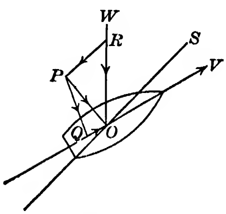


Fig. 38.

(Suggestion. Let WO be the direction of the wind, OS the direction in which the sail is set. Decompose a unit of the wind's force RO into two components, RP parallel to the sail and PO perpendicular to the sail. PR has no effect on the sail, and may therefore be disregarded. The other component PO may again be resolved into two components, namely, PQ perpendicular to the direction of the ship and QO in the direction of the ship. PQ is neutral so far as the forward motion of the ship is concerned, leaving QO as the only part of the wind's force effective in the direction OV.)

13. A person whose eye is at E, 10 ft. above the level of the water PI, observes at I the image of the foot of a pile driven in the water. The horizontal distance of the observer, from the place where the image is formed, is 20 ft., his distance from the pile is 65 ft. What is the length PF of the pile below the surface of the water, the refractive index from air to water being approximately $\frac{4}{3}$.

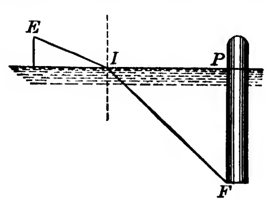


Fig. 39.

Ans. 49.7 ft.

* Index of Refraction $= \dfrac{\text{sine of angle of incidence}}{\text{sine of angle of refraction}}$.

14. Two bodies A, weighing 2 lbs., and B, weighing 3 lbs., are so placed that B is exactly 10 ft. west of A. A moves north and B west, each at the rate of 12 ft. per second. What is the direction and the velocity of their common center of gravity?

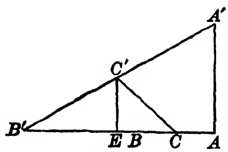


Fig. 40.

Ans. $33° 41'$; 8.65 ft. per sec.

(Suggestion. Locate the center of gravity, C, in two positions, as C and C'. Find EC' and EC, then solve the triangle CEC'.)

15. The arms of a lever are $FA = 2.34$ and $FB = 5.27$ respectively. At the extremity A of the first arm, a force of 5.34 units acts in a direction making an angle of $\alpha = 63° 45'$ with FA produced. What force must be applied at B, the extremity of the second arm, in a direction making an angle $\beta = 51° 15'$ with FB produced, in order that there may be equilibrium?

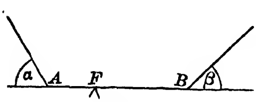


Fig. 41.

16. Six forces, $A = 15$, $B = 6$, $C = 5.7$, $D = 7.9$, $E = 12.3$, $F = 10$, act on the same point and in the same plane.
The angle between A and B is $12° 30'$,
the angle between A and C is $31° 21'$,
the angle between A and D is $47° 46'$,
the angle between A and E is $58° 10'$,
the angle between A and F is $72° 18'$.
Required the resultant force and the direction it makes with A.

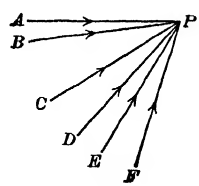


Fig. 42.

Ans. 50.51, $36° 34'$.

(Suggestion. Resolve each force into two components, one along PA, the other in a direction perpendicular to PA. Sum the components along each of these directions separately. The sums are the rectangular components of the required force.)

49. Problems in Navigation. In the following problems it is assumed that the student is acquainted with the divisions of the mariner's compass. On the mariner's compass the total angular space about a point is divided equally into 32 divisions, each of which is called a *point*, that is, a point is equivalent to $\frac{360°}{32} = 11° 15'$.

Each point is divided into two half-points, each half-point into two quarter-points. In the figure below, the names of the 32 points are indicated by their abbreviations. Between north and east the points read:

North by east, north northeast, northeast by north, northeast, northeast by east, east northeast, east by north, and similarly for each of the other quadrants.

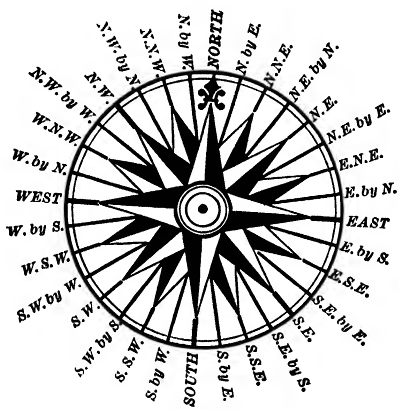


Fig. 43.

In the following problems the surface of the earth is considered a plane and the distances straight lines, not arcs. By a mile is understood a sea mile or knot, which is the length of a minute of arc measured on the earth's equator so that the earth's circumference measures exactly $360 \times 60 = 21{,}600$ sea miles. A sea mile is approximately $1\frac{1}{8}$ common miles.

Definitions. The east and west component of a course, or distance between two points, is called the DEPARTURE of the course or distance, the north and south component is called the difference in LATITUDE, that is, if WN represents any course or distance, and a right triangle WSN is formed, by drawing through W a line east and west, and through N a line north and south, WS is called the departure, and SN the difference in latitude of the course or distance WN.

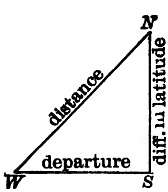


Fig. 44.

In nautical problems difference in latitude is usually expressed in degrees and minutes (1 mile = 1′).

EXERCISE 27

1. A ship sails N.E. by E. at the rate of 8 knots per hour. Find the rate at which it is moving due north.

Ans. $4\frac{4}{9}$ knots per hour.

2. A ship sails S.E. by S. a distance of 578 miles. Find its departure. *Ans.* 321.1 miles.

3. A vessel sails W. by S. until the departure is 315 miles. Find the actual distance sailed. *Ans.* 321.2 miles.

4. A ship sails from latitude $47°30'$ N. on a course N.W. by N. 685 miles. Find the latitude arrived at.

Ans. Diff. in latitude $= 569.6$ miles $= 569.6' = 9°\ 29.6'$.
Required latitude $= 47°\ 30' + 9°\ 29.6' = 56°\ 59.6'$.

5. A ship sails S.W. by S. a distance of 1225 miles. Find the difference in latitude between the first and last positions of the ship and the departure made.

6. A ship sails from latitude $10°\ 24'$ N. and after 30 hrs. reaches latitude $15°\ 26'$ N. Its course was N.N.E. Find the average speed of the ship. *Ans.* 10.9 miles per hr.

7. A ship sails from latitude $35°\ 58'$ N. on a course between S. and E. a distance of 359 miles to a point whose latitude is $32°\ 16'$ N. Find the course of the ship. *Ans.* S. $51°\ 48'$ E.

8. A vessel sails from latitude $5°\ 21'$ S. on a course N.E. by N. a distance of 976 miles. Find the new latitude and the departure.

Ans. $8°\ 11'$ N., 542.3 miles.

9. A steamer bearing W. by N. with a speed of 12 knots has a current setting port * broadside across her track which after 5 hours brings her to an island located 108 miles from her starting point. Find the true course of the ship. *Ans.* N.N.W.

10. One port A is 19 miles due N. of a second port B. Two vessels leave the two ports at the same time, one from B sailing due E. at the rate of 9 knots an hour, the other from A. The vessels meet 5 hours out of port. Determine the speed and the course of the second vessel. *Ans.* 9.77 knots, S. $67°\ 07'$ E.

11. From a "crow's nest" 110 ft. above the water, the angle of depression of a rock just above the water was found to measure

* The left-hand side of the ship as one faces ahead, — opposed to starboard.

15° 36′. Find the distance from the rock to the foot of the mast. *Ans.* 394 ft.

12. A ferry, whose speed in still water is 4 miles per hour, crosses a channel whose current is $3\frac{1}{2}$ miles per hour. How much will she have to "bear up" in order to make the run straight across, and how long will it take her to cross, the channel being 7 miles wide?

13. An observer on board ship notices that the time between the flash of a gun from a fort located N.W. by W. and the report is 5 seconds. After sailing N.E. by N. the gun was heard again, and this time the interval between the flash and the report was 10 seconds. Find the distance sailed and the bearing of the fort from the second position of the ship. *Ans.* 9440 ft., S. 63° 45′ W.
(Assume the velocity of sound to be 1090 ft. per second.)

14. A ship sailing due N. observes two lighthouses in a line due W., and two hours later the bearings of the lighthouses are found to be S. by W. and S.W. by W. respectively. The distance between the lighthouses is known to be 10 miles. Find the rate at which the ship is moving.

15. A man-of-war sailing due N.E. at a uniform speed of 20 knots observes at 9.30 A.M. a fort bearing N.N.W. Twenty-four minutes later the fort is due N.W. Find the distance and bearing of the fort from the ship at 10.15 A.M. *Ans.* 20.54 miles, N. 64° 55′ W.

50. Geographical and Astronomical Problems.

Exercise 28

1. The shadow of a vertical pole 35 ft. high is 51 ft. long. What is the sun's altitude (angle of elevation)? *Ans.* 34° 27.6′.

2. In Fig. 45, let the circle center E represent the earth and the circle center M the moon. The angle PME, formed by the line of centers EM and a line drawn from M tangent to the earth, is known as the moon's equatorial horizontal parallax and measures 57′ 02″. EP, the earth's mean radius, is 3959 miles. Determine EM, the distance of the moon from the earth. *Ans.* By use of S and T table, 238,650 miles.

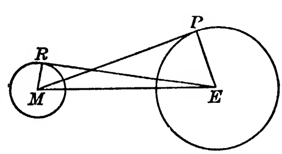


Fig. 45.

3. In Fig. 45, the angle REM, formed by the line of centers ME and a line drawn from E tangent to the moon, is known as the moon's angular semidiameter and measures $15' \, 34''$. Use the result of the last problem and determine RM, the moon's radius.

Ans. By use of S and T table, 1080.6 miles.

4. The sun's equatorial horizontal parallax (see Problem 2) is $8.8''$. The radius of the earth is 3959 miles. Find the distance of the sun from the earth. Also the sun's diameter, the angular semidiameter being $16' \, 02''$. *Ans.* 92,798,000 miles, 865,620 miles.

5. The largest angle between Venus and the sun as seen from the earth is $47° \, 30'$. Using the sun's distance as given in Problem 4, find the distance of Venus from the sun, the orbit of Venus being assumed circular.

6. In Fig. 46, EO represents the earth's radius (= 3959 miles). Find AP, the radius of the arctic circle, latitude $66° \, 32'$. *Ans.* 1577 miles.

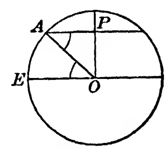


Fig. 46.

7. Prove that lengths of two parallels of latitude are to each other as the cosines of the latitudes.

8. One degree of longitude on the equator is approximately 69.1 miles. Determine the length of a degree of longitude at Seattle, $47° \, 40'$ N. latitude. *Ans.* 46.5 miles.

9. Prove that the lengths of the degrees of longitudes at different latitudes are to each other as the cosines of the latitudes.

10. If one minute of arc of longitude in latitude $60°$ measures 1012.7 yards, how long is the radius of the earth, assuming the earth to be a sphere?

11. A ship sails due W. 540 miles in latitude $36°$ N. What is the difference in longitude between the initial and final positions of the ship? *Ans.* $9° \, 40'$.

12. How high above the Pole would an observer have to be to have the Arctic Circle for his horizon? (Use the data of Problem 6.)

Ans. 357 miles.

13. Beginning at latitude $40°$ N., two consecutive section lines run directly north for a distance of 100 miles. How far apart are they at their northern end? *Ans.* 5166 ft.

14. The shortest shadow cast by a vertical rod 25 ft. long at noon is 21 ft., the longest shadow cast by the same rod at noon is 56 ft. Find the approximate latitude of the place.

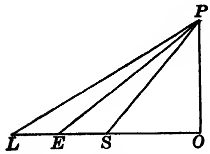


Fig. 47.

Explanation. The shortest shadow, OS, Fig. 47, is cast on June 21, the longest shadow, OL, on December 21. Halfway between these dates the sun will be on the equator; its elevation above the horizon at noon will then be the latitude of the place, that is, angle OPE in the figure, where EP bisects the angle LPS. *Ans.* $52° 59'$.

15. A wall runs east and west; its shadow, measured at right angles to the wall, is 10 ft. wide. The altitude of the sun is $25° 30'$, its azimuth (angular distance west of the south point) is $27° 45'$. Determine the height of the wall.

(Suggestion. Let WE represent the wall, FL the width of the shadow measured at right angles to the wall, MP the direction of the sun. Then angle FMP is the sun's altitude and angle MFL its azimuth. The problem involves two right triangles, namely, MFL and PFM, in which FL, angle MFL and angle PMF are given and PF is to be found.)

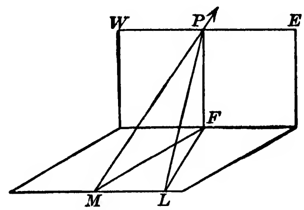


Fig. 48.

16. Each of two observers notices a bright meteor. To the first observer, A, the meteor appeared directly south and at an elevation of 54°. To the second observer, B, stationed 40 miles west of A, the meteor appeared 56° east of the South point. On comparing the times of observation it was ascertained that the same meteor had been observed. Compute the height at which the meteor was seen.

(Suggestion. In Fig. 48 consider L to be A's position, M, B's position, and P the position in which the meteor was observed. PF represents the height of the meteor.)

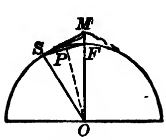


Fig. 49.

17. Find the greatest distance at sea at which a mountain 14,500 ft. high can be seen, the earth being considered a sphere, radius 3960 miles, and the distance sought being the chord joining the point at sea to the foot of the mountain.

Ans. By S and T functions, 137.4 miles.

(Suggestion. The distance sought is $SF = 2 \cdot SO \cdot \sin \frac{1}{2}(SOF)$ and $\cos SOF = \dfrac{OS}{OM}$, but as OS and OM are nearly equal, it is better to use the formula in Article 21 for the determination of angle SOF.)

51. Geometrical Applications. Many geometrical problems can be solved by properly dividing the given figures into right triangles and solving these. Thus, an isosceles triangle is divided into two equal right triangles by drawing a perpendicular from the vertex to the base; any rhombus is divided into four right triangles by its two diagonals; any oblique triangle is equal to the sum or difference of two right triangles, formed by drawing the perpendicular from any vertex to the opposite side; any regular ploygon is divided into as many equal isosceles triangles as the figure has sides, by the lines joining the vertices of the polygon to its center; etc.

EXAMPLE 1. Two sides of a rhombus meet at an angle. The length of one side is a. Find the lengths of the diagonals.

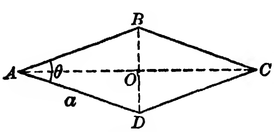


Fig. 50.

Solution. By definition the sides of a rhombus are equal and from geometry it is known that the diagonals intersect at right angles and bisect the angles of the rhombus.

Let $BAD = \theta$ be the given angle, and $AB = AD = a$, the given side.

In the right triangle ABO, two parts, namely, the hypotenuse a and the acute angle $\dfrac{\theta}{2}$ are known, hence we may find AO and OB.

$$AO = a \cos \tfrac{1}{2}\theta, \quad \text{and} \quad BO = a \sin \tfrac{1}{2}\theta,$$

hence

$$AC = 2\,AO = 2\,a \cos \tfrac{1}{2}\theta, \quad \text{and} \quad BD = 2\,BO = 2\,a \sin \tfrac{1}{2}\theta.$$

In particular, if the given side is 15 and the given angle 38°, we have

$$AC = 30 \cos 19° = 28.365,$$

$BD = 30 \sin 19° = 9.768$, by the use of natural functions.

EXAMPLE 2. The radius of a circle is r. To find the perimeter and area of a regular inscribed polygon of n sides.

Solution. By definition the sides of a regular polygon are equal.

Let O represent the center of the circle and AB one side of the inscribed polygon.

The angular magnitude about O is $360°$ and, since there are n sides, angle $AOB = \frac{360°}{n}$.

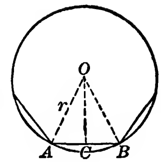


Fig. 51.

Triangle AOB is isosceles, so that if OC is drawn from O perpendicular to AB, it will divide the triangle AOB into two equal right triangles.

In the right triangle AOC, the hypotenuse equals r and the acute angle AOC equals $\frac{360°}{2n} = \frac{180°}{n}$, hence AC and OC can be found.

$$AC = r \sin \frac{180°}{n}, \qquad OC = r \cos \frac{180°}{n},$$

$$AB = 2\,AC = 2\,r \sin \frac{180°}{n},$$

and the perimeter $= n \cdot AB = 2\,nr \sin \frac{180°}{n}$.

Also the area of the triangle $AOB = \frac{1}{2} \cdot AB \cdot OC = r^2 \sin \frac{180°}{n} \cos \frac{180°}{n}$, and the area of the entire polygon $= n$ times the area of triangle AOB

$$= nr^2 \sin \frac{180°}{n} \cos \frac{180°}{n}.$$

In particular, if $r = 95$ and the figure is a heptagon, $n = 7$, and we have

$$\text{Angle } AOC = \frac{180°}{7} = 25°\ 42'\ 51.5''.$$

$$\text{Perimeter} = 14 \cdot 95 \cdot \sin \frac{180°}{7}.$$

$$\text{Area} = 7 \cdot 95^2 \cdot \sin \frac{180°}{7} \cdot \cos \frac{180°}{7}.$$

$\log 14 = 1.14613$

$\log 95 = 1.97772$

$\log \sin \frac{180°}{7} = \underline{9.63737 - 10}$

$\log \text{perimeter} = 2.76122$

$\text{perimeter} = 577.06.$

$\log 7 = 0.84510$

$2 \log 95 = 3.95544$

$\log \sin \frac{180°}{7} = 9.63737 - 10$

$\log \cos \frac{180°}{7} = \underline{9.95471 - 10}$

$\log \text{area} = 4.39262$

$\text{area} = 24696.$

EXAMPLE 3. A right pyramid has for its base a square whose side is $2a$. The angle formed by a face and the base of the pyramid is θ. Determine the altitude, slant height and lateral edge of the pyramid, also the angle made by a lateral edge and the plane of the base, and the angle between the lateral edge and the edge of the base.

Solution. Let V-$ABCD$ represent the pyramid, VO its altitude and VM its slant height. Join M and O. Then $MA = MO = a$, and angle $VMO = \theta$. Further, let

h = altitude VO,
s = slant height MV,
l = lateral edge AV,
α = angle VAO, made by a lateral edge and the plane of the base,
β = angle VAM, made by a lateral edge and an edge of the base.

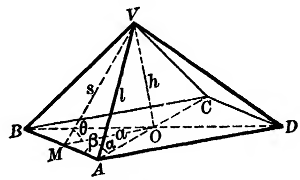


Fig. 52.

In the right triangle VMO, a and θ are given, hence h and s may be found. Solving,

$$h = a \tan\theta, \qquad s = a \sec\theta.$$

In the right triangle VMA, a is given and s has just been found, hence l and β may be found. Solving,

$$\beta = \tan^{-1}\frac{s}{a}, \qquad l = a\sec\beta, \text{ or } = \sqrt{a^2 + s^2}.$$

Finally, from the right triangle VAO,

$$\alpha = \sin^{-1}\frac{h}{l}.$$

As a numerical example take the side of the base equal to 10, and the angle $\theta = 60^\circ$. Then $a = 5$,

and

$$\text{altitude, } h = 5\tan 60^\circ = 5\sqrt{3},$$
$$\text{slant height, } s = 5\sec 60^\circ = 10,$$
$$\text{lateral edge, } l = \sqrt{5^2 + 10^2} = 5\sqrt{5},$$
$$\alpha = \sin^{-1}\sqrt{\tfrac{3}{5}} = 50^\circ\, 46'\, 08'',$$
$$\beta = \tan^{-1} 2 = 63^\circ\, 26'\, 06''.$$

Exercise 29

1. The base of an isosceles triangle is 12 and the angle at the vertex is 48°. Find the altitude of the triangle. *Ans.* 13.48.

2. The chord of a circle is 20 ft. long and the angle at the center subtended by it is 42° 10′. Find the radius of the circle.

Ans. 27.80.

3. The angle between two lines is 50° 21′ 24″ and a circle whose radius is 2380 ft. is tangent to both of them. Find the distance from the intersection of the two lines to the point of tangency, —

(*a*) When the circle lies in the smaller angle,

(*b*) When the circle lies in the larger angle formed by the two lines. *Ans.* 5062.8, 1118.8.

4. The radius of the inscribed circle of an equilateral triangle is r. Find the radius of the circumscribed circle.

5. A chord of a circle subtends at the center an angle of 80° 24′. In the same circle, how large is the angle subtended by a chord half as long? *Ans.* 37° 39.4′.

(Suggestion. Call the length of the chord a.)

6. The side of a regular octagon is 7. Find the area.

Ans. 236.59.

7. The radius of a circle is r; show that a side of a circumscribed regular polygon of n sides is $2r\tan\frac{180^\circ}{n}$.

8. One side of a right triangle is 27.5 and the adjacent acute angle is 54° 38′. Compute the length of the perpendicular from the vertex of the right angle to the hypotenuse, and the segments into which the hypotenuse is divided.

9. Solve the preceding problem, using a for the given side and B for the given angle.

Ans. $p = a\sin B$, $m = a\cos B$, $n = a\sin B\tan B$, where p is the perpendicular, m and n the segments of the hypotenuse, m being the segment adjacent to B.

10. In an oblique triangle two sides and the included angle are given, namely $a = 25.37$, $b = 36.12$, $C = 35^\circ\ 27'$. Find the remaining parts. *Ans.* $A = 43^\circ\ 35.9'$, $B = 100^\circ\ 57.1'$, $c = 21.34$.

(Suggestion. Divide the triangle into two right triangles by drawing a perpendicular from one of the vertices, A or B, to the opposite side.)

11. In an oblique triangle one side and two adjacent angles are given, namely $c = 10$, $A = 60°$, $B = 75°$. Find the remaining parts.

Ans. $C = 45°$, $a = 5\sqrt{6}$, $b = 5(1 + \sqrt{3})$.

(Suggestion. Divide the triangle into two right triangles by drawing a perpendicular from B to the side opposite.)

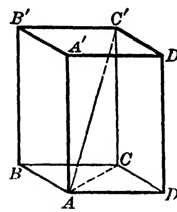


Fig. 53.

12. A regular parallelopiped has for its base a rectangle whose dimensions are $AB = 8$, $AD = 10$, and its altitude $AA' = 15$. Find the angles which the diagonal AC' makes with AD, with AC and with AB.

13. A right pyramid has $2a$ for an edge of the regular hexagon which forms its base and an altitude equal to a. Find the angles which a lateral edge makes with an edge of the base, with the plane of the base, and the angle which a lateral face makes with the plane of the base.

Ans. $\tan^{-1} 2 = 63° \, 26'$, $\tan^{-1} \frac{1}{2} = 26° \, 34'$, $\tan^{-1} \frac{1}{3}\sqrt{3} = 30°$.

14. Verify trigonometrically the following practical rule for inscribing a regular pentagon in a circle:

Let O be the center of the circle, OA and OC two perpendicular radii. Bisect OA in M. Take MR equal to MC. With C as center and CR as a radius, draw an arc cutting the circle in P. Join C and P. PC will be the side of the pentagon.

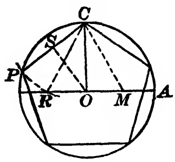


Fig. 54.

(Suggestion. Assume the radius of length r.
From triangle OCM find $CM = RM$.
From triangle COR find $CR = CP$.
From triangle COS ($SC = \frac{1}{2} PC$) find angle SOC, which should be $36°$.)

52. Oblique Triangles Solved by Right Triangles. Every oblique triangle may be solved by decomposing it into right triangles. This is done by drawing a perpendicular from one of the vertices of the triangle to the opposite side. In three of the four cases the perpendicular can be so chosen that two of the given parts become parts of one of the right triangles. This triangle having been solved, two parts of the other right triangle become known. The second triangle may now be solved, and with this all the parts of the original triangle become known. The fourth case (given the three sides)

requires a somewhat different method. We will take up each case separately.

Case I. *Given one side and two adjacent angles*, as b, C, A.

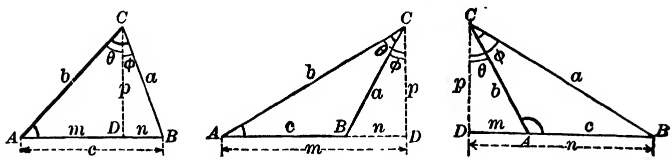


Fig. 55.

Let ABC represent the triangle. From A or C, say C, draw a perpendicular CD to the opposite side, or opposite side produced.

Let $CD = p$, $AD = m$, $DB = n$, angle $ACD = \theta$, angle $BCD = \phi$. Three different figures may arise, —

> the left-hand figure, when A and B are both acute,
> the middle figure, when A is acute and B obtuse,
> the right-hand figure, when A is obtuse.

In the right triangle ACD, b and angle CAD are known, hence p, m, and θ may be found.

From C and θ, ϕ may be found.

Having found ϕ and p, we know two parts of the right triangle BCD, hence a and n and the angle CBD may be found.

Knowing m and n, c may be found.

To check the answers, we repeat the solution, drawing the perpendicular from A instead of from C.

Case II. *Given two sides and the angle opposite one of these sides*, as a, b, A.

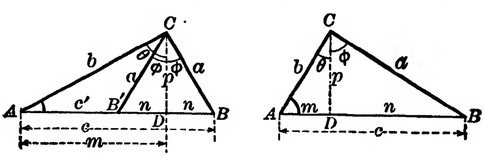


Fig. 56.

In this case draw the perpendicular from that vertex which lies between the two given sides a and b.

If $a < b$, two different triangles exist which have the given parts, as in Fig. 56, left, if $a \geqq b$, only one triangle exists.

In the right triangle CAD, b and A are given, hence p, m and θ (= angle ACD) may be found.

Next consider the right triangle BCD. p and a are known, hence n and ϕ (= angle BCD) may be found.

Finally,

$$AB = c = m + n, \quad C = \theta + \phi, \quad B = 180° - (A + C).$$

The second solution, if there is one, as in the figure to the left, is given by

$$AB' = c' = m - n, \; ACB' = C' = \theta - \phi, \; AB'C = B' = 180° - (A + C').$$

Check. From triangle ACD, $m = b \cos A$,

From triangle CDB, $n = a \cos B$,

and since $m + n = c$, we must have

$$b \cos A + a \cos B = c.$$

Case III. *Given two sides and an included angle*, as b, c, A.

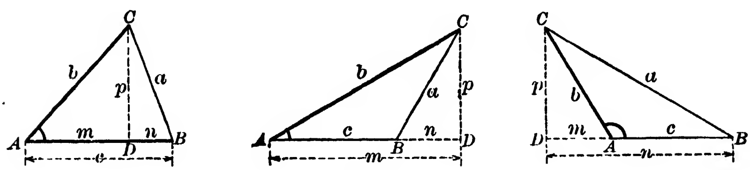


Fig. 57.

Draw the perpendicular from B or C, say C.

In the right triangle ACD, b and angle CAD are known, hence p and m may be found.

Knowing m and c, n may be found.

Having found p and n, we know two parts of the right triangle BCD, hence a and angle CBD may be found.

Finally, angle $ACB = 180° - (A + \text{angle } ABC)$.

Here, as in Case I, three different figures are possible, according as

A and B are both acute (left-hand figure),

A acute and B obtuse (middle figure),

A obtuse (right-hand figure),

but the above analysis applies to each figure alike.

A check is obtained by repeating the solution with the perpendicular drawn from B.

Case IV. *Given the three sides, a, b, c.*

In the first three cases we were able so to choose the vertex, from which the perpendicular was drawn, that one of the right triangles contained two of the given parts. In the present case this is not possible. The apparent difficulty is easily overcome as follows, —

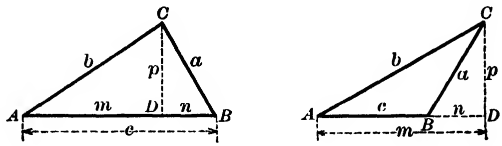


Fig. 58.

In either of the figures 58 we have

from the triangle CAD, $p^2 = b^2 - m^2$,

from the triangle CDB, $p^2 = a^2 - n^2$,

hence $b^2 - m^2 = a^2 - n^2$,

from which $b^2 - a^2 = m^2 - n^2 = (m + n)(m - n)$. (1)

We now have,

in the left figure, $m + n = c$, in the right figure, $m - n = c$,

and from (1),

$$m - n = \frac{b^2 - a^2}{m + n} = \frac{b^2 - a^2}{c} = \frac{(b + a)(b - a)}{c},$$

$$m + n = \frac{b^2 - a^2}{m - n} = \frac{b^2 - a^2}{c} = \frac{(b + a)(b - a)}{c},$$

from which, since a, b, c are given,

$$m - n \qquad \text{or} \qquad m + n$$

may be found.

We then have in either case,

$$\frac{(m + n) + (m - n)}{2} = m,$$

$$\frac{(m + n) - (m - n)}{2} = n.$$

We now know two parts in each of the right triangles CAD and CDB, and from these the angles A and B may be computed.

As a check the solution may be repeated with the perpendicular drawn from one of the other vertices.

EXAMPLE. Given $a = 45.652$, $b = 62.735$, $c = 51.238$; to find the angles.

Solution.
$$m - n = \frac{(b+a)(b-a)}{c}$$

$$\begin{array}{ll} b + a = 62.735 + 45.652 = 108.387, & \log(b+a) = 2.03498 \\ b - a = 62.735 - 45.652 = 17.083, & \log(b-a) = 1.23257 \\ c = 51.238. & \operatorname{colog} c = \underline{8.29041} \\ & \log(m-n) = 1.55796 \\ & m - n = 36.138\,^* \end{array}$$

$$m = \frac{(m+n)+(m-n)}{2} = \frac{51.238 + 36.138}{2} = 43.688,$$

$$n = \frac{(m+n)-(m-n)}{2} = \frac{51.238 - 36.138}{2} = 7.550.$$

From the right triangle CAD,

$$\frac{m}{b} = \cos A.$$

$$\begin{aligned} \log m &= 1.64036 \\ \operatorname{colog} b &= \underline{8.20249} \\ \log \cos A &= 9.84285 - 10 \\ A &= 45^\circ\, 51'\, 46'', \end{aligned}$$

From the right triangle CDB,

$$\frac{n}{a} = \cos B.$$

$$\begin{aligned} \log n &= 0.87795 \\ \operatorname{colog} a &= \underline{8.34054} \\ \log \cos B &= 9.21849 - 10 \\ B &= 80^\circ\, 28'\, 49'', \end{aligned}$$

$$C = 180^\circ - (A + B) = 53^\circ\, 39'\, 24''.$$

Check. Interchanging b and c in the above formula, we find

$$\begin{array}{ll} c + a = 51.238 + 45.652 = 96.890, & \log(c+a) = 1.98628 \\ c - a = 51.338 - 45.652 = 5.586, & \log(c-a) = 0.74710 \\ b = 62.735. & \operatorname{colog} b = \underline{8.20249} \\ & \log(m-n) = 0.93587 \\ & m - n = 8.6272. \end{array}$$

$$m = \frac{(m+n)+(m-n)}{2} = \frac{62.735 + 8.6272}{2} = 35.681,$$

$$n = \frac{(m+n)-(m-n)}{2} = \frac{62.735 - 8.6272}{2} = 27.054.$$

* If this result were greater than c, we would have the right-hand figure in Fig. 58, and we should have taken $m + n = \dfrac{(b+a)\,(b-a)}{c}$.

$$\frac{m}{c} = \cos A. \qquad\qquad \frac{m}{a} = \cos C.$$

$$\begin{array}{rlrl} \log m &= 1.55243 & \log n &= 1.43223 \\ \operatorname{colog} c &= \underline{8.29041} & \operatorname{colog} a &= \underline{8.34055} \\ \log\cos A &= 9.84284 & \log\cos C &= 9.77278 \\ A &= 45^\circ\, 51'\, 51'', & C &= 53^\circ\, 39'\, 24'', \end{array}$$

$$B = 180^\circ - (A + C) = 80^\circ\, 28'\, 45''.^*$$

Exercise 30

Only a few triangles are given here for solution by the method of right triangles, for soon we shall study a better method, by means of which the computation can in most cases be shortened.

1. Given $a = 342.56$, $b = 125.72$, $C = 37^\circ\, 42'\, 24''$.
   Ans. $A = 124^\circ\, 44'\, 28''$, $B = 17^\circ\, 33'\, 08''$, $c = 254.97$.

2. Given $b = 134.5$, $c = 235.2$, $A = 127^\circ\, 36.3'$.
   Ans. $a = 334.7$, $B = 18^\circ\, 33.9'$.

3. $A = 25^\circ\, 25'\, 25''$, $B = 50^\circ\, 50'\, 50''$, $c = 278.98$.
   Ans. $C = 103^\circ\, 43'\, 45''$, $a = 123.29$, $b = 222.70$.

4. $C = 127^\circ\, 36.5'$, $A = 28^\circ\, 31.3'$, $b = 312.9$.
   Ans. $c = 612.55$, $a = 369.22$, $B = 23^\circ\, 52.2'$.

5. $a = 630.50$, $b = 527.39$, $A = 65^\circ\, 37'\, 12''$.
   Ans. $B = 49^\circ\, 37'\, 38''$, $C = 64^\circ\, 45'\, 10''$, $c = 626.13$.

6. $b = 1825$, $c = 1563$, $B = 22^\circ\, 13.7'$. Find the remaining parts.
   Ans. $C = 14^\circ\, 54.8'$, $A = 142^\circ\, 51.5'$, $a = 2913$.

7. $a = 3.537$, $b = 6.667$, $c = 5.001$. Find the remaining parts.

8. $a = 4$, $b = 5$, $c = 6$. Find the remaining parts.
   Ans. $A = 41^\circ\, 24.6'$, $B = 55^\circ\, 46.3'$, $C = 82^\circ\, 49.1'$.

* When checking five-place distances and angles expressed to seconds obtained from five-place tables, the results will generally be found to agree only to four places for distances and to the nearest tenth of a minute (6″) for angles. This is because, as we have already observed, the fifth place of a number and the seconds of an angle obtained from a five-place table are not necessarily accurate. When cut down to the nearest tenth of a minute, the results of the two computations in the above example agree, each giving

$$A = 45^\circ\, 51.8', \quad B = 80^\circ\, 28.8', \quad C = 53^\circ\, 39.4'.$$

The solution may therefore be assumed to be correct.

9. Find the ratio between the sides of a triangle whose angles are, $A = 50°$, $B = 60°$, $C = 70°$.

Ans. $a : b : c = 0.7660 : 0.8660 : 0.9397$.

10. In a quadrilateral $ABCD$, Fig. 59, the following parts are known:

$AB = 673$, $BC = 589$, $CD = 223$,
angle $B = 105° 06'$, angle $C = 127° 38'$.

It is required to find the length of AD to the nearest unit. *Ans.* 1017.

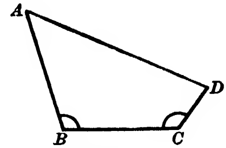


Fig. 59.

CHAPTER VI

FUNCTIONS OF AN OBTUSE ANGLE

Reason for a New Definition. In a right triangle no angle can exceed 90°, but when the triangle is oblique one of its angles may be obtuse, that is, one of its angles may have any value between 90° and 180°. In order to solve oblique triangles in the simplest way possible, we must define the trigonometric functions for obtuse angles. This is best done by means of the conception of rectangular coördinates.

53. Rectangular Coördinates. Let $X'X$ and $Y'Y$ be two lines, indefinite in length, intersecting at right angles at O. The two lines divide their plane into four parts, known as the first, second, third and fourth quadrants respectively, as indicated by the numerals I, II, III, IV, in Fig. 60.

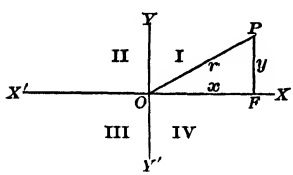


Fig. 60.

Let P be any point in the plane of the lines $X'X$ and $Y'Y$, and let PF be the perpendicular drawn from P to $X'X$. Join O and P. OP, the distance of P from O, is always positive and is designated by r.

OF, generally represented by x, is called the *abscissa* of the point P. It is considered positive if P is to the right, negative if to the left of $Y'Y$.

FP, generally represented by y, is called the *ordinate* of the point P. It is considered positive if P is above, negative if below $X'X$.

Considered together, OF or x and FP or y are known as the rectangular *coördinates* of the point P.

$X'X$ and $Y'Y$ are called the *coördinate axes*, or axes of reference; $X'X$ is the x-axis, or *axis of abscissas*; $Y'Y$ is the y-axis, or *axis of ordinates*; O is called the *origin*.

The abscissa of a point is always written first and the ordinate second. Thus, by the point (a, b), we understand the point for

which $x = a$, $y = b$, that is, the point whose abscissa is a and whose ordinate is b.

From what has been said it is plain that:

In the first quadrant, x is positive, y is positive, r is positive;
in the second quadrant, x is negative, y is positive, r is positive;
in the third quadrant, x is negative, y is negative, r is positive;
in the fourth quadrant, x is positive, y is negative, r is positive.

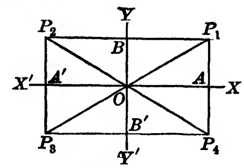


Fig. 61.

For every point on the x-axis, $y = 0$;
for every point on the y-axis, $x = 0$;
for the origin, x and y are each 0.

Thus, in Fig. 61,
if for $\quad P_1,\ x = 4,\ y = 3;$

then for P_2, $x = -4, y = 3$; then for B, $x = 0, y = 3$;
for P_3, $x = -4, y = -3$; for A', $x = -4, y = 0$;
for P_4, $x = 4, y = -3$; for B', $x = 0, y = -3$;
for A, $x = 4, y = 0$; for O, $x = 0, y = 0$;

but each of the distances OP_1, OP_2, OP_3, OP_4 equals $\sqrt{3^2 + 4^2} = \sqrt{25} = 5$.

54. Definition of the Trigonometric Functions of Any Angle Less than 180°.

Let angle $XOB = \theta$ represent any angle less than 180°. Take O for an origin, OX for the positive x-axis, and draw OY perpendicular to OX. Then OB will be in the first or second quadrant according as θ is acute or obtuse. Now take any point P on OB and denote by r the distance of this point from the origin, and by x and y the rectangular coördinates of this point with reference to OX and OY as axes. The trigonometric functions of θ are then defined as follows, —

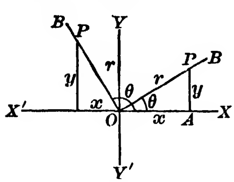


Fig. 62.

$$\sin\theta = \frac{y}{r} = \frac{\text{ordinate}}{\text{distance}}, \qquad \csc\theta = \frac{1}{\sin\theta},$$

$$\cos\theta = \frac{x}{r} = \frac{\text{abscissa}}{\text{distance}}, \qquad \sec\theta = \frac{1}{\cos\theta},$$

$$\tan\theta = \frac{y}{x} = \frac{\text{ordinate}}{\text{abscissa}}, \qquad \cot\theta = \frac{1}{\tan\theta}.$$

It will be observed that when the angle θ is acute, these definitions agree with those given in Art. 7; for x, y and r are the base, altitude and hypotenuse respectively of the right triangle which we there used in defining the trigonometric functions of an acute angle.

55. The Signs of the Functions of an Obtuse Angle. In the first quadrant, x and y, as well as r, are positive. Hence all the functions are positive for an angle in the first quadrant. In the second quadrant, x is negative, y and r are positive, hence the ratios $\frac{x}{r}$ and $\frac{y}{x}$ are negative, while $\frac{y}{r}$ remains positive, that is, the cosine and tangent and their reciprocals are negative, while the sine and its reciprocal are positive.

The functions of an obtuse angle are all negative, except the sine and its reciprocal, which are positive.

56. Fundamental Relations. Of the six fundamental relations, —

$$\sin\theta\cdot\csc\theta = 1, \qquad \sin^2\theta + \cos^2\theta = 1,$$
$$\cos\theta\cdot\sec\theta = 1, \qquad \tan^2\theta + 1 = \sec^2\theta,$$
$$\tan\theta\cdot\cot\theta = 1, \qquad \cot^2\theta + 1 = \csc^2\theta,$$

the first three rest upon the definitions of the cosecant, secant and cotangent, and hold, therefore, whether θ is acute or obtuse, and the last three depend upon the relation $x^2 + y^2 = r^2$, which is true whether x is positive or negative. These six fundamental relations hold, therefore, for the functions of obtuse angles as well as for the functions of acute angles.

Also,

$$\tan\theta = \frac{y}{x} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{\sin\theta}{\cos\theta},$$ whether x is positive or negative, hence

the two relations, —

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta},$$

also hold true when θ is an obtuse angle.

57. Functions of Supplementary Angles. Let $XOP = \theta$ be any angle less than 180°, and draw OP' so as to make angle $P'OX'$ equal to angle XOP. Then angle $XOP' = 180° - \theta$.

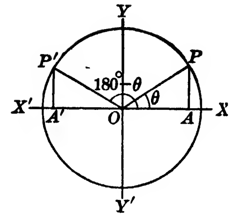


Fig. 63.

If OP and OP' are taken equal, the two triangles OAP and $OA'P'$ will be geometrically equal, and we have

$$\sin(180° - \theta) = \frac{A'P'}{OP'} = \frac{AP}{OP} = \sin\theta,$$

$$\cos(180° - \theta) = \frac{OA'}{OP'} = \frac{-OA}{OP} = -\frac{OA}{OP} = -\cos\theta,$$

$$\tan(180° - \theta) = \frac{A'P'}{OA'} = \frac{AP}{-OA} = -\frac{AP}{OA} = -\tan\theta,$$

$$\csc(180° - \theta) = \frac{1}{\sin(180° - \theta)} = \frac{1}{\sin\theta} = \csc\theta,$$

$$\sec(180° - \theta) = \frac{1}{\cos(180° - \theta)} = \frac{1}{-\cos\theta} = -\sec\theta,$$

$$\cot(180° - \theta) = \frac{1}{\tan(180° - \theta)} = \frac{1}{-\tan\theta} = -\cot\theta.$$

By comparing our results we observe that the signs on the right are those of the functions in the second quadrant, hence it appears that, —

Any function of $(180° - \theta)$ *equals plus or minus the same function of* θ, *the sign being that of the function in the second quadrant.*

The rule just given enables us to express the functions of an obtuse angle in terms of the functions of an acute angle, thus:

$$\sin 116° = \sin(180° - 64°) = \sin 64°,$$

$$\cos 116° = \cos(180° - 64°) = -\cos 64°,$$

$$\tan 116° = \tan(180° - 64°) = -\tan 64°,$$

$$\csc 116° = \csc(180° - 64°) = \csc 64°,$$

$$\sec 116° = \sec(180° - 64°) = -\sec 64°,$$

$$\cot 116° = \cot(180° - 64°) = -\cot 64°.$$

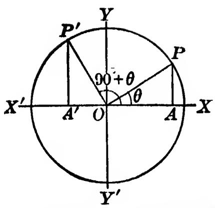


Fig. 64.

58. Functions of $(90° + \theta)$. Let angle $XOP = \theta$ be any acute angle and draw OP' perpendicular to OP. Then angle $XOP' = 90° + \theta$. If OP' is taken equal to OP, the triangles AOP and $A'OP'$ are geometrically equal, and we have

$$\sin(90° + \theta) = \frac{A'P'}{OP'} = \frac{OA}{OP} = \cos\theta,$$

$$\cos(90° + \theta) = \frac{OA'}{OP'} = \frac{-AP}{OP} = -\frac{AP}{OP} = -\sin\theta,$$

$$\tan(90° + \theta) = \frac{A'P'}{OA'} = \frac{OA}{-AP} = -\frac{OA}{AP} = -\cot\theta,$$

$$\csc(90° + \theta) = \frac{1}{\sin(90° + \theta)} = \frac{1}{\cos\theta} = \sec\theta,$$

$$\sec(90° + \theta) = \frac{1}{\cos(90° + \theta)} = \frac{1}{-\sin\theta} = -\csc\theta,$$

$$\cot(90° + \theta) = \frac{1}{\tan(90° + \theta)} = \frac{1}{-\cot\theta} = -\tan\theta.$$

Again the signs on the right are the signs of the functions in the second quadrant, hence,—

Any function of $(90° + \theta)$ is equal to plus or minus the corresponding cofunction of θ, the sign being that of the function in the second quadrant.

This rule, like that of the preceding article, enables us to express the functions of an obtuse angle in terms of the functions of an angle less than 90°. Thus,

$$\sin 116° = \sin(90° + 26°) = \cos 26°,$$
$$\cos 116° = \cos(90° + 26°) = -\sin 26°,$$
$$\tan 116° = \tan(90° + 26°) = -\cot 26°, \text{ etc.}$$

59. Functions of 180°. If in Fig. 62, θ is taken equal to 180°, OP must coincide with OX', the abscissa of P will be $-r$ and its ordinate zero, hence

$$\sin 180° = \frac{0}{r} = 0, \qquad \csc 180° = \frac{1}{0} = \infty,$$

$$\cos 180° = \frac{-r}{r} = -1, \qquad \sec 180° = \frac{1}{-1} = -1,$$

$$\tan 180° = \frac{0}{-r} = -0, \qquad \cot 180° = \frac{1}{-0} = -\infty.$$

The results in the last line need some explanation. The tangent and cotangent of every obtuse angle is negative, hence their limiting values, as the angle approaches 180°, is negative. Numerically these limiting values are 0 and ∞, the minus signs merely indicate that these values have been approached through a succession of negative magnitudes.

60. Angles Corresponding to a Given Function. Since the cosine, secant, tangent and cotangent of every obtuse angle is negative, we can tell whether the angle corresponding to one of these functions is obtuse or acute by noting the algebraic sign of the function. But the sine and cosecant are positive whether the angle is acute or obtuse. In fact, since the sines and cosecants of supplementary angles are equal in every respect, there will always be two angles, one acute and the other obtuse, which will correspond equally well to a given sine or cosecant. This is expressed by saying that the angle corresponding to a given sine or cosecant is ambiguous.

EXAMPLES. If $\cos\theta = \frac{1}{2}$, θ must equal 60°,

and if $\cos\theta = -\frac{1}{2}$, θ equals the supplement of 60°, or 120°.

If $\tan\theta = 1$, θ must equal 45°,

and if $\tan\theta = -1$, θ equals the supplement of 45°, or 135°.

If $\sin\theta = \frac{1}{2}$, θ may be either 30° or its supplement, 150°.

EXERCISE 31

1. Locate the points whose coördinates are $x = 8, y = 6$; $x = -8, y = -6$; $x = -8, y = 6$; $x = 8, y = -6$; and in each case compute the distance of the point from the origin.

2. Locate the points $x = 5, y = 0$; $x = 0, y = 5$; $x = -1, y = -1$; $x = 0, y = -1$.

3. The distance of a point from the origin is 2, and its abscissa is 1; find its ordinate and locate the point.

Ans. Two solutions, $y = \pm\sqrt{3}$.

4. The distance of a point from the origin is 10 and its ordinate is $\sqrt{50}$; find the abscissa and locate the point.

5. Make out a table containing the sine, cosine and tangent of each of the angles 120°, 135°, 150°.

6. Construct the angles, having given the following functions: $\cos A = -\frac{3}{8}$, $\tan B = -3$, $\sin \theta = \frac{1}{3}$.

7. Express the following functions in terms of functions of the supplementary angles: $\sin 115°$, $\tan 165°$, $\cos 125°$, $\cot 100°$, $\sec 170°$, $\sin 145°$, $\cos 136°$, $\tan 95°$.

8. Express in terms of an angle less than 45° the following:

$$\sin 95°,\ \cos 120°,\ \tan 100°,\ \sec 114°,\ \sin 125°.$$

9. Express in two ways in terms of an acute angle, first by means of the rule of Art. 57, second by means of the rule of Art. 58, each of the following:

$$\sin 127°\,35'\,13'',\ \cos 157°\,54'\,36'',\ \tan 140°\,11'\,25.3''.$$

Which of the two methods is the easier?

10. Use the table of natural functions to find the following functions: $\sin 112°\,30'$, $\cos 156°\,25'$, $\tan 162°\,50'$, $\sin 105°$, $\cos 175°\,10'$, $\tan 126°\,14'$.

11. θ being any obtuse angle, prove the following relations:

$$\begin{aligned}
\sin(\theta - 90°) &= -\cos\theta, & \csc(\theta - 90°) &= -\sec\theta,\\
\cos(\theta - 90°) &= \sin\theta, & \sec(\theta - 90°) &= \csc\theta,\\
\tan(\theta - 90°) &= -\cot\theta, & \cot(\theta - 90°) &= -\tan\theta.
\end{aligned}$$

61. Review.

1. (*a*) What is meant by the logarithm of a number to a given base a? (*b*) Show that $\log ab = \log a + \log b$, $\log a^b = b \log a$. (*c*) What is the logarithm of 1? (*d*) What is meant by a cologarithm? (*e*) Given $\log 4 = 0.60206$; find $\log 16$, $\log 2$, $\log \frac{1}{4}$.

2. (*a*) What is meant by the common logarithm of a number? (*b*) Give from memory the common logarithms of 10, 100, 0.1, 0.01, 10^r, $\frac{1}{10^n}$, $\sqrt{10}$, $\sqrt[3]{10}$, $\sqrt[3]{100}$. (*c*) What is meant by the characteristic of a common logarithm? What by the mantissa? (*d*) Give the characteristics of the logarithms of the following numbers: 15, 153, 6.23, 0.05, 0.0105. (*e*) Give the rule for the characteristic of a number greater than 1; of a decimal fraction less than 1.

3. Explain how a table of common logarithms might be constructed by extracting square roots only. Find the number whose logarithm is $\frac{3}{8}$, without consulting the table.

4. Prove that $\log_a N = \dfrac{\log_b N}{\log_b a}$; also show that $\log_b a \cdot \log_a b = 1$.

5. What is meant by the principle of proportional parts? Read again Art. 32.

6. Work Problems 13 and 14, Exercise 18.

7. What is meant by an exponential equation? Solve Problem 18, Exercise 18.

8. (*a*) Read again Art. 44. (*b*) What accuracy is called for in the angle of a triangle, when the sides are given correct to three places? To four places? (*c*) How accurately can a number be determined with the aid of a five-place table of logarithms? An angle? (*d*) When should a six-place table be used? When a seven-place table?

9. (*a*) How many degrees constitute a point of a mariner's compass? (*b*) How many degrees in the angle between N. and N.E. by N.? Between N.E. by E. and E.N.E.? (*c*) What direction is opposite to the direction S.E. by S.?

10. Explain how to solve each of the four cases of oblique triangles by means of right triangles.

11. (*a*) What is meant by the rectangular coördinates of a point? (*b*) The line joining the origin to a point P is 5 units in length and makes an angle of $30°$ with the positive direction of the x-axis. What are the rectangular coördinates of the point P? (*c*) What are the coördinates of the point P, if OP makes an angle of $150°$ with OX?

12. (*a*) Define the sine, cosine and tangent of an obtuse angle. (*b*) Prove that $\sin(180° - A) = \sin A$, $\cos(180° - A) = -\cos A$, $\tan(90° + A) = -\cot A$. (*c*) Complete the equations

$$\tan(180° - A) = \quad, \sin(90° + A) = \quad, \cos(90° + A) = \quad.$$

13. (*a*) Find $\sin 123°$, $\cos 136°$, $\tan 105° 30'$. (*b*) Find x in each of the equations: $\sin x = 0.3423$, $\cos x = -0.9061$, $\tan x = -0.0913$, x being in each case the angle of a plane triangle.

CHAPTER VII

PROPERTIES OF TRIANGLES

In this chapter we shall develop certain properties of triangles which will enable us to compute, from a sufficient number of given parts, the remaining sides and angles, the area and other magnitudes related to the triangle. The principal applications of the results developed in this chapter will be treated in a separate chapter.

62. The Law of Sines.

(*a*) *First proof.* Let ABC be any plane triangle. Draw the perpendicular h from one of the vertices C of the triangle to the opposite side AB (Fig. 65), or AB produced (Fig. 66).

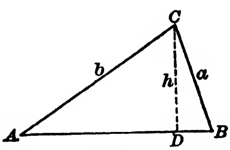


Fig. 65.

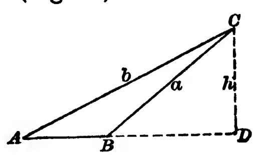


Fig. 66.

In Fig. 65,
in the right triangle ACD

$$h = b \sin A,$$

and in the right triangle BCD

$$h = a \sin B.$$

In Fig. 66,
in the right triangle ACD

$$h = b \sin A,$$

and in the right triangle BCD

$$h = a \sin (180^\circ - B) = a \sin B.$$

Hence, whether the triangle is acute or obtuse, we have

$$h = b \sin A = a \sin B,$$

or

$$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

Similarly, by drawing a perpendicular from B to the opposite side, or the opposite side produced, we obtain

$$\frac{a}{\sin A} = \frac{c}{\sin C},$$

so that we may write

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \tag{1}$$

Equation (1) may be otherwise written thus,—

$$a : b : c = \sin A : \sin B : \sin C. \tag{2}$$

Equation (1) or (2) embodies what is known as the *Law of Sines*, which states that,—

In any triangle the sides are proportional to the sines of the opposite angles.

(*b*) *Second proof.* The Law of Sines may be proven in another way, which at the same time brings out the meaning of the ratios in equation (1).

Circumscribe a circle about the triangle ABC and denote by D the diameter BA', drawn through one of the vertices, as B. Join A' and C. $A'BC$ is a right triangle (Why?), and therefore

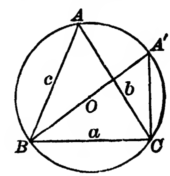


Fig. 67.

$$\frac{a}{D} = \sin A', \text{ or } D = \frac{a}{\sin A'}.$$

But angle A' = angle A (angles inscribed in the same arc are equal), hence

$$D = \frac{a}{\sin A'} = \frac{a}{\sin A},$$

and similarly

$$D = \frac{b}{\sin B}, \qquad D = \frac{c}{\sin C},$$

from which

$$D = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \tag{3}$$

that is,—

The ratio of any side of a triangle to the sine of the angle opposite is numerically equal to the diameter of the circumscribed circle.

63. Projection Theorem.

In Fig. 65,	In Fig. 66,
$AD = b \cos A,$	$AD = b \cos A,$
$DB = a \cos B.$	$BD = a \cos (180° - B) = -a \cos B.$

Moreover,

$c = AB = AD + DB.$	$c = AB = AD - BD.$

Substituting for AD, DB, and BD their values, we have

$$c = b\cos A + a\cos B. \qquad\qquad c = b\cos A - (-a\cos B) = b\cos A + a\cos B.$$

Hence, whether the triangle is acute or obtuse, we have

Similarly,*
$$\left.\begin{aligned} c &= a\cos B + b\cos A.\\ a &= b\cos C + c\cos B,\\ b &= c\cos A + a\cos C.\end{aligned}\right\} \qquad (4)$$

We may consider the line DB (Fig. 66) the negative of BD, that is, $DB = -BD$. In that case

$c = AB = AD - BD = AD - (-DB) = AD + DB$, just as in Fig. 65.

$AD = b\cos A$ is called the *projection* of AC on AB,

$DB = a\cos B$ is called the projection of CB on AB,

so that the relations (4) may be stated thus, —

In any triangle, each side is equal to the algebraic sum of the projections of the other two sides upon it.

64. The Law of Cosines.

(*a*) *First proof.*

In Fig. 65,

$$\begin{aligned} b^2 &= h^2 + \overline{AD}^2\\ h^2 &= a^2 - \overline{DB}^2\\ \overline{AD}^2 &= (c - DB)^2\\ &= c^2 - 2c\cdot DB + \overline{DB}^2.\end{aligned}$$

In Fig. 66,

$$\begin{aligned} b^2 &= h^2 + \overline{AD}^2\\ h^2 &= a^2 - \overline{BD}^2\\ \overline{AD}^2 &= (c + BD)^2\\ &= c^2 + 2c\cdot BD + \overline{BD}^2.\end{aligned}$$

Substituting in the first equation the values of h and AD from the second and third, we have

$$\begin{aligned} b^2 &= a^2 - \overline{DB}^2 + c^2 - 2c\cdot DB + \overline{DB}^2\\ &= a^2 + c^2 - 2c\cdot DB.\end{aligned} \qquad\qquad \begin{aligned} b^2 &= a^2 - \overline{BD}^2 + c^2 + 2c\cdot BD + \overline{BD}^2\\ &= a^2 + c^2 + 2c\cdot BD.\end{aligned}$$

Now from the figure

$$DB = a\cos B. \qquad\qquad BD = a\cos(180^\circ - B) = -a\cos B.$$

* The second formula may be obtained from the first by replacing a by b, b by c, c by a, and A by B, B by C, and C (should it occur) by A. In the same manner the third formula may be obtained from the second, and the first from the third. That is, if any one of these formulas is given, the other two can be supplied by *cyclic substitution.*

Substituting these values in the equation just preceding, we obtain in either case

$$\left.\begin{aligned} b^2 &= c^2 + a^2 - 2\,ca\cos B. \\ c^2 &= a^2 + b^2 - 2\,ab\cos C, \\ a^2 &= b^2 + c^2 - 2\,bc\cos A. \end{aligned}\right\} \qquad (5)$$

Similarly,

These formulas embody the so-called

Law of Cosines: In any triangle, the square on any side is equal to the sum of the squares on the other two sides diminished by twice the product of those two sides times the cosine of the included angle.

(*b*) *Second proof.* The law of cosines may be proved even more easily than above by the aid of the projection formulas. We need only to multiply the first of the formulas (4) by c, to add a times the second and to subtract b times the third. The result is

$$\begin{aligned} c^2 + a^2 - b^2 &= c\,(a\cos B + b\cos A) + a\,(b\cos C + c\cos B) \\ &\quad - b\,(c\cos A + a\cos C) \\ &= 2\,ac\cos B, \end{aligned}$$

from which

$$b^2 = c^2 + a^2 - 2\,ca\cos B.$$

65. Arithmetic Solution of Triangles. The law of sines and the law of cosines are sufficient to solve each of the four cases of oblique triangles.

I. If one side a and two angles are given, the third angle is found immediately from the relation $A + B + C = 180°$. The remaining sides may then be found from the law of sines, thus, —

$$b = \frac{a\sin B}{\sin A}, \qquad c = \frac{a\sin C}{\sin A}.$$

II. If two sides b and c and the angle opposite one of them, say B, is given, the third side a may be found by the law of cosines; for, solving the first of equations (5), considering a as the unknown quantity, we find

$$a = c\cos B \pm \sqrt{b^2 - c^2\sin^2 B}.$$

This gives two values for a, as it should, for we know that this case has in general two solutions.

Having found the third side, the angles A and C may now be found from the law of sines, thus:

$$\sin A = \frac{a}{b} \sin B, \qquad \sin C = \frac{c}{b} \sin B.$$

III. If two sides b and c and the included angle A are given, the third side a may be found from the law of cosines, for the third equation (5) gives

$$a = \sqrt{b^2 + c^2 - 2\,bc \cos A}.$$

The sides and one angle being known, the law of sines will give the other angles.

IV. If the three sides are given, the three angles may be found from the law of cosines. Thus, to find A, we have from the third of the equations (5),

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}.$$

While the law of sines and the law of cosines are theoretically all that is necessary to solve triangles, the law of cosines, which would have to be used in two out of the four cases, is not adapted to logarithmetic computation. The numerical work necessary to solve triangles will be greatly shortened by the use of other formulas which we will develop in the following articles.

Exercise 32

1. Apply the law of sines to a right triangle and reduce the resulting equations to their simplest form.

2. Apply the law of cosines, $a^2 = b^2 + c^2 - 2\,bc \cos A$, to the cases when the angle $A = 0°, 90°, 180°$.

Ans. $A = 0°$, $a^2 = b^2 + c^2 - 2\,bc = (b - c)^2$.
$A = 90°$, $a^2 = b^2 + c^2$.
$A = 180°$, $a^2 = b^2 + c^2 + 2\,bc = (b + c)^2$.

3. Apply the projection theorem to the case $A = 90°$; to the case $A = B$.

Solve the following problems without the aid of logarithms, —

4. Given $A = 35°$, $B = 75°$, $a = 7$; find b and c.

Ans. $b = 11.8$, $c = 11.5$.

5. Given $A = 65°$, $b = 10$, $a = 15$; find B and c.

Ans. $B = 37° \ 10'$, $c = 16.2$.

6. Given $A = 16°$, $b = 15$, $a = 6$; find the remaining parts.

Ans. $B = 43° \ 33.5'$, $C = 120° \ 26.5'$, $c = 18.76$,

or $B' = 136° \ 26.5'$, $C' = 27° \ 33.5'$, $c' = 10.07$.

7. Given $a = 150$, $b = 200$, $C = 27° \ 30'$; find c.

Ans. $c = 96.3$.

8. Given $a = 2$, $b = 3$, $c = 4$; find the angles.

Ans. $A = 28° \ 57.3'$, $B = 46° \ 34.1'$, $c = 104° \ 28.6'$.

9. By means of the law of sines prove that the bisector of an angle of any triangle divides the opposite side into segments proportional to the adjacent sides.

10. Derive the law of sines from the law of cosines.

(Suggestion. Form the ratio $\dfrac{\sqrt{1 - \cos^2 A}}{\sqrt{1 - \cos^2 B}} = \dfrac{\sin A}{\sin B}$ and show that it is equal to $\dfrac{a}{b}$.)

66. The Law of Tangents.

(*a*) *First proof.* Let ABC be any triangle. With a vertex C as a center and b, the shorter of the sides adjacent to C, as a radius, draw a circle cutting BC in P and BC produced in Q. Draw AP and AQ. Triangles ACP and ACQ are isosceles and QAP is a right angle (Why?). Denote the whole angle at A by w, and the three parts by x, y, z, as indicated in the figure, then angle $APC = x$ and angle $AQC = z$ (Why?). Also

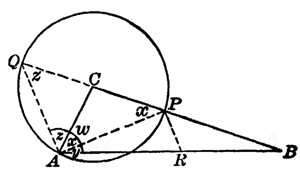


Fig. 68.

$$x + y = A, \quad x - y = B, \quad x + z = 90°, \quad x + y + z = w.$$

Solving these equations for x, y, z and w, we obtain

$$x = \tfrac{1}{2}(A + B), \quad y = \tfrac{1}{2}(A - B), \quad z = 90° - \tfrac{1}{2}(A + B),$$
$$w = 90° + \tfrac{1}{2}(A - B).$$

Now apply the law of sines to each of the triangles APB and AQB, thus, —

$$\frac{AB}{BP} = \frac{\sin(180 - x)}{\sin y} = \frac{\sin x}{\sin y}, \quad \text{and} \quad \frac{AB}{BQ} = \frac{\sin z}{\sin w},$$

or

$$\left.\begin{aligned}
\frac{c}{a-b} &= \frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)}, & \frac{c}{a+b} &= \frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)}. \\
&\text{Similarly,} \\
\frac{a}{b-c} &= \frac{\sin\frac{1}{2}(B+C)}{\sin\frac{1}{2}(B-C)}, & \frac{a}{b+c} &= \frac{\cos\frac{1}{2}(B+C)}{\cos\frac{1}{2}(B-C)}, \\
\frac{b}{c-a} &= \frac{\sin\frac{1}{2}(C+A)}{\sin\frac{1}{2}(C-A)}, & \frac{b}{c+a} &= \frac{\cos\frac{1}{2}(C+A)}{\cos\frac{1}{2}(C-A)}.
\end{aligned}\right\} \quad (6)$$

Dividing the first of each pair of equations by the second gives

$$\left.\begin{aligned}
\frac{a+b}{a-b} &= \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}, \quad \frac{b+c}{b-c} = \frac{\tan\frac{1}{2}(B+C)}{\tan\frac{1}{2}(B-C)}, \\
\frac{c+a}{c-a} &= \frac{\tan\frac{1}{2}(C+A)}{\tan\frac{1}{2}(C-A)}.
\end{aligned}\right\} \quad (7)$$

Formulas (7) embody the

Law of tangents: In any triangle, the sum of two sides is to their difference as the tangent of half the sum of the angles opposite is to the tangent of half their difference.

The formulas (6), which we shall have occasion to use hereafter, we shall refer to by the name of *Double Formulas.**

(*b*) *Second Proof.* The law of tangents can be proven more easily without the intervention of the double formulas. In Fig. 68 draw PR parallel to QA, then angle APR = angle $QAP = 90°$. From the similar triangles BQA and BPR, we have

$$\frac{BQ}{BP} = \frac{AQ}{RP} = \frac{\dfrac{AQ}{AP}}{\dfrac{RP}{AP}},$$

but
$$BQ = a + b, \quad BP = a - b,$$

and
$$\frac{AQ}{AP} = \tan APQ = \tan x = \tan\tfrac{1}{2}(A+B),$$

$$\frac{RP}{AP} = \tan RAP = \tan y = \tan\tfrac{1}{2}(A-B),$$

* Also called Mollweide's formulas, after the German astronomer (1774–1825) who introduced their use. The cosine form of these formulas appears in Newton's *Arithmetica universalis* (1707).

hence

$$\frac{a+b}{a-b}=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}.$$

The law of tangents and double formulas are adapted to the logarithmic solution of the third case of oblique triangles, that is, when two sides and the included angle are given. Suppose the given parts are a, b and C. The different steps in the solution are as follows:

1. $\frac{1}{2}(A+B)$ is found from the relation $A+B=180^\circ-C$.

2. $\frac{1}{2}(A-B)$ is found from the law of tangents, formula (7), 1st. equation.

3. Adding and subtracting the results of 1 and 2 we have
$\frac{1}{2}(A+B)+\frac{1}{2}(A-B)=A, \qquad \frac{1}{2}(A+B)-\frac{1}{2}(A-B)=B.$

4. c is found from the double formula (6).

Having found A and B, c could have been determined from the law of sines, thus, —

$$c=\frac{a\sin C}{\sin A}, \quad \text{or} \quad c=\frac{b\sin C}{\sin B},$$

but this would require us to look up three new logarithms, namely, those of

$$a, \sin C, \sin A, \quad \text{or} \quad b, \sin C, \sin B.$$

while the double formula requires but two new logarithms, those of

$$\sin\tfrac{1}{2}(A+B), \sin\tfrac{1}{2}(A-B), \quad \text{or} \quad \cos\tfrac{1}{2}(A+B), \cos\tfrac{1}{2}(A-B),$$

and these may be taken out at the same time and the same opening of the table with the logarithms of $\tan\frac{1}{2}(A+B)$ and $\tan\frac{1}{2}(A-B)$.

67. Formulas for the Area of a Triangle.

(*a*) *In terms of the base c and the altitude h.* If c represents the base of the triangle, Figs. 65, 66, h its altitude and T its area, then by elementary geometry

$$T=\tfrac{1}{2}ch. \tag{1}$$

(*b*) *In terms of two sides b and c and the included angle A.* From the right triangle, Fig. 65 or 66, we have $h=b\sin A$. This value of h substituted in (1) gives

$$T=\tfrac{1}{2}bc\sin A. \tag{2}$$

(*c*) *In terms of one side c and the angles A, B, C.* By the law of sines,

$$b = \frac{c \sin B}{\sin C}.$$

Substituting this value of b in (2) we obtain

$$T = \frac{c^2}{2} \cdot \frac{\sin A \sin B}{\sin C}. \tag{3}$$

(*d*) *In terms of the three sides a, b, c.* It is shown in plane geometry that

$$T = \sqrt{s(s-a)(s-b)(s-c)},^* \tag{4}$$

where $s = \frac{1}{2}(a + b + c)$, that is, s equals half the sum of the three sides.

(*e*) *In terms of s and the radius k of the inscribed circle.*

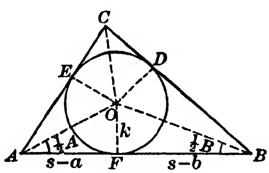


Fig. 69.

Let O be the center of the inscribed circle, Fig. 69. Join O to the vertices of the triangle, and draw the radii of the inscribed circle to the points of tangency. These radii will be perpendicular to the respective sides.

$$\text{Area of triangle } BOC = \tfrac{1}{2} ka,$$
$$\text{Area of triangle } COA = \tfrac{1}{2} kb,$$
$$\text{Area of triangle } AOB = \tfrac{1}{2} kc.$$

Adding,

$$T = \tfrac{1}{2} ka + \tfrac{1}{2} kb + \tfrac{1}{2} kc = \tfrac{1}{2} k(a + b + c),$$

or

$$T = ks, \text{ where } s = \tfrac{1}{2}(a + b + c). \tag{5}$$

If the three sides are known separately, (5) enables us to find the radius of the inscribed circle of a triangle, for we have

$$k = \frac{T}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \tag{6}$$

on substituting the value of T from (4).

* This formula is known as Hero's formula for the area of a triangle, after Hero of Alexandria (1st century B.C.), who, so far as we know, was the first to prove and apply this remarkable formula.

68. Functions of Half the Angles in Terms of the Sides.

In Fig. 69,

$$\begin{aligned} AF &= AE \\ BD &= BF \\ DC &= EC \end{aligned}$$

Adding, $$AF + (BD + DC) = BF + (AE + EC),$$
and since the sum of the six segments equals $a + b + c = 2s$, we have

$$AF + (BD + DC) = s, \quad BF + (AE + EC) = s,$$

or $$AF + a = s, \qquad BF + b = s,$$

from which

$$AF = s - a, \qquad BF = s - b. \qquad (1)$$

Also, since the lines AO and BO bisect the angles A and B respectively, we have

$$\tan\frac{A}{2} = \frac{k}{AF}, \qquad \tan\frac{B}{2} = \frac{k}{BF}.$$

Substituting for AF and BF their values from (1), we have

$$\left.\begin{aligned} \tan\frac{A}{2} &= \frac{k}{s-a}, \\ \tan\frac{B}{2} &= \frac{k}{s-b}, \\ \text{similarly}\quad \tan\frac{C}{2} &= \frac{k}{s-c}, \end{aligned}\right\} \text{where } k = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}. \qquad (2)$$

Again

$$\overline{AO}^2 = \overline{OF}^2 + \overline{AF}^2 = k^2 + (s-a)^2$$
$$= \frac{(s-a)(s-b)(s-c) + s(s-a)^2}{s} = \frac{bc(s-a)}{s}. \qquad (3)$$

$$\sin\frac{A}{2} = \frac{OF}{AO} = \frac{k}{AO}, \qquad \cos\frac{A}{2} = \frac{AF}{AO} = \frac{s-a}{AO}.$$

Substituting for AO its value from (3) and reducing the result gives,—

$$\left.\begin{aligned} \sin\frac{A}{2} &= \sqrt{\frac{(s-b)(s-c)}{bc}}, & \cos\frac{A}{2} &= \sqrt{\frac{s(s-a)}{bc}}. \\ \text{Similarly}\quad \sin\frac{B}{2} &= \sqrt{\frac{(s-c)(s-a)}{ca}}, & \cos\frac{B}{2} &= \sqrt{\frac{s(s-b)}{ca}}, \\ \sin\frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{ab}}, & \cos\frac{C}{2} &= \sqrt{\frac{s(s-c)}{ab}}. \end{aligned}\right\} \qquad (4)$$

In each of the radicals the positive root is to be taken, since each of the angles $\frac{A}{2}$, $\frac{B}{2}$, $\frac{C}{2}$, is necessarily less than 90°.

The formulas (2) and (4) are adapted for the logarithmetic computation of the angles of a triangle when the sides are given. For the quantities s, $s - a$, $s - b$, $s - c$, can easily be found, and with these known, the right member of each formula involves only multiplications, divisions and the extraction of square roots.

In general, the angles may be found from either the sine, cosine or tangent formulas, but since an angle near 90° cannot be accurately found from its sine, nor a very small angle from its cosine, the sine formulas are to be avoided when the angle is greater than 45° and the cosine formulas when the angle is less than 45°. When all the angles are required, the tangent formulas (2) should be used, since they require but four logarithms to be taken from the tables, that is, the logarithms of s, $s - a$, $s - b$ and $s - c$, while the sine and cosine formulas (4) require in addition the logarithms of a, b and c.

Exercise 33

1. Express in words each of the rules for the area of a triangle in Art. 67.

2. From each of the formulas (2) and (3) in Art. 67, write down two others by a cyclic advance of letters.

3. By comparing the formulas (2) and (4) in Art. 67 show that

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

Why could not the angles be found by this formula as well as by (2) or (4), Art. 68?

4. By comparing the expressions for $\sin A$ in Problem 3 with the expressions for $\sin\frac{A}{2}$, $\cos\frac{A}{2}$ in (4), Art. 68, show that

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}.$$

5. By forming the ratio of $\sin A : \sin B$, using the values given in Problem 3, show that

$$\sin A : \sin B = a : b.$$

This constitutes another proof of the law of sines.

6. From (3), Art. 62, we have $\sin A = \frac{a}{D}$, where D is the diameter of the circumscribed circle. By substituting this value for $\sin A$ in (2), Art. 67 show that

$$T = \frac{abc}{2D}.$$

7. From (5), Art. 67, we have for the radius of the inscribed circle

$$k = \frac{T}{s}.$$

By using the relation (Fig. 70)

$$ABC = ABO + ACO - BCO$$

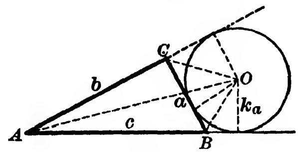


Fig. 70.

show that

$$k_a = \frac{T}{s-a}, \quad k_b = \frac{T}{s-b}, \quad k_c = \frac{T}{s-c},$$

where k_a, k_b, k_c are the radii of the escribed circles, touching the sides a, b, c respectively, externally.

8. Prove that

$$\frac{1}{k} = \frac{1}{k_a} + \frac{1}{k_b} + \frac{1}{k_c}.$$

9. Prove that $\quad k_a + k_b + k_c - k = 2D.$

10. Let $ABCD$ (Fig. 71) be an inscribed quadrilateral, a, b, c, d its sides and Q its area. Show that

$$Q = \tfrac{1}{2}(ad + bc)\sin A. \qquad (a)$$

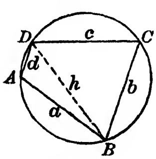


Fig. 71.

Also by comparing the two expressions for the diagonal

$$h^2 = a^2 + d^2 - 2ad\cos A = b^2 + c^2 - 2bc\cos C$$

show that

$$\cos A = \frac{a^2 - b^2 - c^2 + d^2}{2(ad + bc)}, \qquad (b)$$

and by substituting this value in $\sin^2 A = 1 - \cos^2 A$, show that

$$\sin A = \frac{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}{ad + bc}, \qquad (c)$$

where $\qquad s = \tfrac{1}{2}(a + b + c + d).$

By substituting (c) in (a), show that

$$Q = \sqrt{(s-a)(s-b)(s-c)(s-d)}. \qquad (d)$$

11. It was shown in Art. 65, II, that if two sides b, c, of a triangle ABC, and the angle B opposite one of these sides, are given, the third side a may be found from the relation

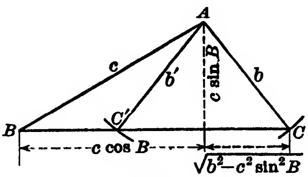


Fig. 72.

$$a = c\cos B \pm \sqrt{b^2 - c^2\sin^2 B}.$$

Interpret the result geometrically.

Ans. The two terms on the right are respectively the distances from B and C to the foot of the perpendiculars drawn from A to BC or BC produced.

CHAPTER VIII

SOLUTION OF OBLIQUE TRIANGLES

69. Solution of Oblique Triangles. In the present chapter we shall illustrate the computation of each case of oblique triangles by a numerical example. Since five-place tables are used in the computation, the results are given to only five significant figures and the angles to the nearest second. In every case a check has been applied to the results obtained. Such a check is to be looked upon as an essential step in the solution, since no computation, no matter by whom, can be relied upon if it has not been checked. Instead of the analytic checks here given, graphic checks are often resorted to in practice, but such checks, while more easily applied, are of course open to errors of construction and are therefore unsatisfactory, except as checks against gross errors.

The solutions given below are arranged in a form which may serve as a model to beginners. It is customary for computers to make out a complete schedule of work (as is illustrated in the first case below) before referring to the tables, so that when the tables are once opened no writing remains to be done except that of filling in the numbers taken from the tables.

70. Case I. Given two angles and one side, as *A*, *B* and *c*.

(1) To find C, apply the relation $A + B + C = 180°$.

(2) To find a and b, apply the law of sines.

(3) To check, use the double formula.

EXAMPLE 1.

Given

$A = 71° \ 13' \ 30''$,
$B = 40° \ 34' \ 15''$,
$c = 236.54$.

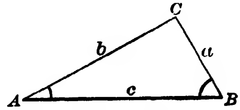


Fig. 73.

Required

$a = 241.18$,
$b = 165.68$,
$C = 68° \ 12' \ 15''$.

Schedule of Work.

(1) To find C. $\quad C = 180° - (A + B) = 68° \ 12' \ 15''$.

(2) To find a and b. By the law of sines, —

$$\frac{a}{c} = \frac{\sin A}{\sin C} \text{ or } a = \frac{c \sin A}{\sin C}. \qquad \frac{b}{c} = \frac{\sin B}{\sin C} \text{ or } b = \frac{c \sin B}{\sin C}.$$

$$\begin{array}{rl} \log c = & \\ \log \sin A = & \\ \operatorname{colog} \sin C = & \underline{\hspace{6em}} \\ \log a = & \\ a = & \end{array} \qquad \begin{array}{rl} \log c = & \\ \log \sin B = & \\ \operatorname{colog} \sin C = & \underline{\hspace{6em}} \\ \log b = & \\ b = & \end{array}$$

(3) Check.* By the double formula (6), left,

$$\frac{c}{a-b} = \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} \quad \text{or} \quad a - b = \frac{c \sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)},$$

$$\tfrac{1}{2}(A-B) = 15^\circ\, 19'\, 37'', \qquad \tfrac{1}{2}(A+B) = 55^\circ\, 53'\, 52''.$$

$$\begin{array}{rl} \log c = & \\ \log \sin \frac{1}{2}(A-B) = & \\ \operatorname{colog} \sin \frac{1}{2}(A+B) = & \underline{\hspace{6em}} \\ \log (a-b) = & \end{array} \qquad a - b =$$

If the computation is correct this value of $a - b$ must agree with the value of $a - b$ obtained from (2) above.

Having completed a schedule as above, we now turn to the tables and complete the solution by filling in the missing numbers as follows:

Solution.

$$\begin{array}{rl} \log c = & 2.37388 \\ \log \sin A = & 9.97626 - 10 \\ \operatorname{colog} \sin C = & \underline{0.03221 \hspace{3em}} \\ \log a = & 2.38235 \\ a = & 241.18. \end{array} \qquad \begin{array}{rl} \log c = & 2.37388 \\ \log \sin B = & 9.81318 - 10 \\ \operatorname{colog} \sin C = & \underline{0.03221 \hspace{3em}} \\ \log b = & 2.21927 \\ b = & 165.68. \end{array}$$

Check.

$$\begin{array}{rl} \log c = & 2.37388 \\ \log \sin \frac{1}{2}(A-B) = & 9.42214 - 10 \\ \operatorname{colog} \sin \frac{1}{2}(A+B) = & \underline{0.08195 \hspace{3em}} \\ \log (a-b) = & 1.87797, \end{array} \qquad a - b = 75.50.$$

Beginners will do well to follow the above form. Expert computers save the repetition of recurring numbers such as log c in the above example by employing a more compact arrangement. For instance, the above solution and check can be put in the following

* Some authors check by the law of sines, $a : b = \sin A : \sin B$, but this check is unreliable, for it fails to detect an error in either c or log sin C.

Compact Arrangement

	Solution.	Check.
$c = 236.54$	2.37388	2.37388
$A = 71^\circ\ 13'\ 30''$	9.97626	
$B = 40^\circ\ 34'\ 15''$	9.81318	
$C = 68^\circ\ 12'\ 15''$	0.03221	
$\frac{1}{2}(A - B) = 15^\circ\ 19'\ 37''$		9.42214
$\frac{1}{2}(A + B) = 55^\circ\ 53'\ 52''$		0.08195
$a = 241.18$	2.38235	
$b = 165.68$	2.21927	
$a - b = 75.50$		1.87797

EXAMPLE 2. Find the area of the triangle given in Example 1.

Solution. Since one side and two angles are given, we use formula (3), Art. 67,

$$T = \frac{c^2 \sin A \sin B}{2 \sin C}$$

$$\begin{aligned} \log c &= 2.37388 \\ \log c^2 &= 2 \log c = 4.74776 \\ \log \sin A &= 9.97626 - 10 \\ \log \sin B &= 9.81318 - 10 \\ \operatorname{colog} \sin C &= 0.03221 \\ \operatorname{colog} 2 &= \underline{9.69897 - 10} \\ \log T &= 4.26838, \qquad T = 18552. \end{aligned}$$

EXERCISE 34

The student must check his results when no answer is given.

1. Given $A = 46^\circ\ 36'$, $B = 54^\circ\ 18'$, $c = 479$.
   Find $a = 354.4$, $b = 396.1$, $C = 79^\circ\ 06'$.
2. Given $A = 79^\circ\ 59'$, $B = 44^\circ\ 41'$, $a = 79.5$.
   Find $b = 56.8$, $c = 66.4$, $C = 55^\circ\ 20'$.
3. Given $A = 54^\circ\ 34'$, $B = 43^\circ\ 56'$, $c = 67.9$. Find a, b, C.
4. Given $A = 69^\circ\ 30.2'$, $B = 66^\circ\ 39.4'$, $c = 438.3$.
   Find $a = 592.7$, $b = 581.0$, $C = 43^\circ\ 50.4'$.
5. Given $A = 29^\circ\ 41.2'$, $B = 37^\circ\ 50.4'$, $a = 32.84$.
   Find $b = 40.68$, $c = 61.27$, $C = 112^\circ\ 28.4'$.
6. Given $B = 78^\circ\ 45.6'$, $C = 63^\circ\ 32.9'$, $a = 8.875$. Find b, c, A.
7. Given $A = 64^\circ\ 56'\ 18''$, $B = 47^\circ\ 29'\ 11''$, $c = 913.45$.
   Find $a = 895.14$, $b = 728.40$, $C = 67^\circ\ 34'\ 31''$.

8. Given $B = 48° 24' 15''$, $C = 31° 13' 00''$, $c = 926.74$.
   Find $b = 1337.2$, $a = 1758.9$, $A = 100° 22' 45''$.

9. Find the radius of the circumscribed circle in 8.
   Ans. $R = 894.06$.

10. Find the area of the triangle in 7. *Ans.* Area = 301,360.

11. Find the area and the radius of the circumscribed circle in 2. Check by using the relation in Problem 6, Exercise 33.

71. Case II. Given two sides and the angle opposite one of them, as a, b, A.

(1) To find B, apply the law of sines.
(2) To find C, apply the relation $A + B + C = 180°$.
(3) To find c, apply the law of sines.
(4) To check, apply the double formula or the law of tangents.

When an angle is determined from its sine, as the angle B above, it admits of two values which are supplements of each other. Whether one or the other or both of these values are to be used depends on the conditions imposed by the problem. By constructing the triangle graphically, it is seen that various cases may arise depending on the relations between the given parts a, b and A.

Construct angle BAC equal to the given angle A, making CA equal to b, and from C as a center, with a radius equal to a, draw a circle.

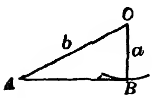
If a is less than the perpendicular distance from C to AB, the circle will not cut the line AB. In this case it is impossible to construct a triangle having the given parts, that is, the given data are inconsistent.

If a is equal to the perpendicular distance from C to AB, the circle will be tangent to the line AB at B (Fig. 74), and the resulting triangle will have a right angle at B. Now the perpendicular distance from C to AB is $b \sin A$, hence in this case $a = b \sin A$.

If a is greater than the perpendicular distance from C to AB but less than b, the circle will cut AB in two points B and B' (Fig. 75), and there are two solutions, namely, the triangle ABC and the triangle $AB'C$.

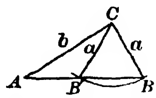
If a is equal or greater than b, the circle will cut the line AB in a single point B (Fig. 76), the second point of intersection falling on or to the left of A. In this case there will be but one solution.

So far we have assumed the given angle A to be acute. If A is obtuse, a must be greater than b (since A is greater than B, and in any triangle the greater angle is opposite the greater side). In this case there can be but one solution. (Fig. 77.)



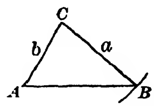
$a = b \sin A.$

Fig. 74.



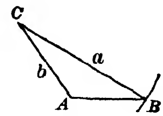
$b \sin A < a < b.$

Fig. 75.



$a \gtreqqless b.$

Fig. 76.



$a > b.$

Fig. 77.

If we disregard the case in which the triangle is right-angled (Fig. 74) as not properly constituting a case of oblique triangles, we have the following simple test for the number of possible solutions,—

$a \geqq b$, one solution,

$a < b$, two solutions.

EXAMPLE 1.

Given

$a = 345.46,$

$b = 531.75,$

$A = 26° 47' 32''.$

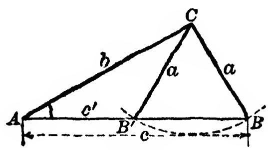


Fig. 78.

Required

$B = 43° 56' 00'',$

$B' = 136° 04' 00'',$

$C = 109° 16' 28'',$

$C' = 17° 08' 28'',$

$c = 723.45.$

$c' = 225.88.$

Solution. $a < b$, hence there are two solutions.

(1) To find B and B'. By the law of sines,—

$$\frac{a}{b} = \frac{\sin A}{\sin B}, \quad \text{or} \quad \sin B = \frac{b \sin A}{a}.$$

$$\begin{aligned} \log b &= 2.72571 \\ \log \sin A &= 9.65394 - 10 \\ \operatorname{colog} a &= 7.46160 - 10 \\ \hline \log \sin B &= 9.84125 - 10 \\ B &= 43° 56' 00''. \qquad B' = 136° 04' 00''. \end{aligned}$$

(2) To find C and C'.

$C = 180° - (A + B) = 109° 16' 28''. \quad C' = 180° - (A + B') = 17° 08' 28''.$

(3) To find c and c'. By the law of sines $\dfrac{c}{a} = \dfrac{\sin C}{\sin A}$, or

$$c = \frac{a \sin C}{\sin A}. \qquad\qquad c' = \frac{a \sin C'}{\sin A}.$$

$$\begin{aligned} \log a &= 2.53840 \\ \log \sin C &= 9.97495 - 10 \\ \operatorname{colog} \sin A &= \underline{0.34606} \\ \log c &= 2.85941 \\ c &= 723.45. \end{aligned} \qquad\qquad \begin{aligned} \log a &= 2.53840 \\ \log \sin C' &= 9.46942 - 10 \\ \operatorname{colog} \sin A &= \underline{0.34606} \\ \log c' &= 2.35388 \\ c' &= 225.88. \end{aligned}$$

(4) Check. By the double formula,—

$$\frac{c}{b-a} = \frac{\sin \frac{1}{2}(B+A)}{\sin \frac{1}{2}(B-A)}$$

or

$$\begin{aligned} c \sin \tfrac{1}{2}(B-A) &= (b-a) \sin \tfrac{1}{2}(B+A), \\ \tfrac{1}{2}(B+A) &= 35^\circ\, 21'\, 46'', \\ \tfrac{1}{2}(B-A) &= 8^\circ\, 34'\, 14'', \\ c' \sin \tfrac{1}{2}(B'-A) &= (b-a) \sin \tfrac{1}{2}(B'+A), \\ \tfrac{1}{2}(B'+A) &= 81^\circ\, 25'\, 46'', \\ \tfrac{1}{2}(B'-A) &= 54^\circ\, 38'\, 14'', \\ b-a &= 186.29. \end{aligned}$$

$$\begin{aligned} \log c &= 2.85941 \\ \log \sin \tfrac{1}{2}(B-A) &= \underline{9.17327 - 10} \\ & \ 2.03268 \end{aligned} \qquad \begin{aligned} \log (b-a) &= 2.27019 \\ \log \sin \tfrac{1}{2}(B+A) &= \underline{9.76249 - 10} \\ & \ 2.03268 \end{aligned}$$

$$\begin{aligned} \log c' &= 2.35388 \\ \log \sin \tfrac{1}{2}(B'-A) &= \underline{9.91143} - 10 \\ & \ 2.26531 \end{aligned} \qquad \begin{aligned} \log (b-a) &= 2.27019 \\ \log \sin \tfrac{1}{2}(B'+A) &= \underline{9.99513} - 10 \\ & \ 2.26532 \end{aligned}$$

Compact Arrangement

	Solution.		Check.
$b = 531.75$	log 2.72571		
$A = 26^\circ\, 47'\, 32''$	log 9.65394	colog 0.34606	
$a = 345.46$	colog $\underline{7.46160}$	log 2.53850	
$B = 43^\circ\, 56'\, 00''$	log 9.84125		
$C = 109^\circ\, 16'\, 28''$		log $\underline{9.97496}$	
$c = 723.45$		log 2.85941	log 2.85941
$\frac{1}{2}(B-A) = 8^\circ\, 34'\, 46''$			log 9.17327
$\frac{1}{2}(B+A) = 35^\circ\, 21'\, 46''$			colog 0.23751
$b - a = 186.29$			colog $\underline{7.72981}$
			0.00000

$B' = 136°\ 04'\ 00''$		
$C' = 17°\ 08'\ 28''$	$\log \underline{9.46942}$	
$c' = 225.88$	$\log 2.35388$	$\log 2.35388$
$\frac{1}{2}(B' - A) = 54°38'14''$		$\log 9.91143$
$\frac{1}{2}(B' + A) = 81°25'46''$		colog 0.00487
$b - a = 186.29$		colog $\underline{7.72981}$
		9.99999

Exercise 35

Solve the following triangles:

1. $a = 840,\quad b = 485,\quad A = 21°\ 31'.$
   Ans. $B = 12°\ 14',\quad C = 146°\ 15',\quad c = 1272.$
2. $a = 41.4,\quad b = 52.8,\quad A = 40°\ 19'.$
   Ans. $B = 55°36'\quad C = 84°\ 05',\quad c = 63.6.$
   $B' = 124°\ 24',\quad C' = 15°\ 17',\quad c' = 16.9.$
3. $a = 3.25,\quad b = 2.57,\quad A = 32°\ 54'.$
4. $a = 242,\quad b = 767,\quad B = 36°\ 53'.$
   Ans. $A = 10°\ 55',\quad C = 132°\ 12',\quad c = 947.$
5. $a = 91.97,\quad b = 93.99,\quad B = 120°\ 35'.$
   Ans. $A = 57°\ 23.7',\quad C = 2°\ 01.3',\quad c = 3.85.$
6. $b = 978.7,\quad c = 871.6,\quad C = 38°\ 14.2'.$
   Ans. $B = 44°\ 01.5',\quad A = 97°\ 44.3',\quad a = 1395.$
   $B' = 135°\ 58.5',\quad A' = 5°\ 47.3',\quad a' = 142.$
7. $b = 678.5,\quad c = 423.1,\quad C = 53°\ 23.4'.$
8. $a = 48.134,\quad b = 35.826,\quad A = 36°\ 24'\ 00''.$
   Ans. $B = 26°\ 12'\ 38'',\quad C = 117°\ 23'\ 22'',\quad c = 72.022.$
9. $b = 216.45,\quad c = 177.01,\quad C = 35°\ 36'\ 20''.$
   Ans. $B = 45°\ 23'\ 28'',\quad A = 99°\ 00'\ 12'',\quad a = 300.29.$
   $B' = 134°\ 36'\ 32'',\quad A' = 9°\ 47'\ 08'',\quad a' = 51.67.$
10. $b = 14.332,\quad c = 13.617,\quad C = 45°\ 23'\ 54''.$
11. $a = 342.6,\quad b = 745.9,\quad A = 43°\ 35.6'.$
    Ans. Impossible.

72. Case III. Given two sides and the included angle, as *a*, *b*, *C*.

To find A and B, we first find $\frac{1}{2}(A + B)$ and $\frac{1}{2}(A - B)$.

(1) To find $\frac{1}{2}(A + B)$, apply the relation $A + B + C = 180°$.

(2) To find $\frac{1}{2}(A - B)$, apply the law of tangents, Art. 66.

(3) $A = \frac{1}{2}(A+B) + \frac{1}{2}(A-B)$, $B = \frac{1}{2}(A+B) - \frac{1}{2}(A-B)$.

(4) To find c, apply the law of sines.

(5) To check, apply one of the double formulas.

EXAMPLE I.

Given

$a = 12.346$,
$b = 5.7213$,
$C = 65° 30' 10''$.

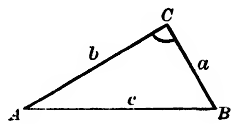


Fig. 79.

Required

$A = 86° 55' 57''$,
$B = 27° 33' 53''$,
$c = 11.250$.

Solution.

(1) To find $\frac{1}{2}(A+B)$. $\frac{1}{2}(A+B) = \frac{1}{2}(180° - C) = 57° 14' 55''$.

(2) To find $\frac{1}{2}(A-B)$. By the law of tangents

$$\frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}, \quad \text{or} \quad \tan\tfrac{1}{2}(A-B) = \frac{a-b}{a+b}\tan\tfrac{1}{2}(A+B).$$

$a - b = 6.6247$, $\qquad a + b = 18.0673$.

$$\begin{aligned}
\log(a-b) &= 0.82117\\
\operatorname{colog}(a+b) &= 8.74310 - 10\\
\log\tan\tfrac{1}{2}(A+B) &= 0.19162\\
\hline
\log\tan\tfrac{1}{2}(A-B) &= 9.75589 - 10\\
\tfrac{1}{2}(A-B) &= 29° 41' 02''.
\end{aligned}$$

(3) To find A and B.

$$A = \tfrac{1}{2}(A+B) + \tfrac{1}{2}(A-B) = 86° 55' 57'',$$
$$B = \tfrac{1}{2}(A+B) - \tfrac{1}{2}(A-B) = 27° 33' 53''.$$

(4) To find c. From the law of sines we have either

$$c = a\frac{\sin C}{\sin A}, \quad \text{or} \quad c = b\frac{\sin C}{\sin B},$$

but since A is an angle near $90°$ it is preferable to find c from the second expression.

$$\begin{aligned}
\log b &= 0.75749\\
\log\sin C &= 9.95903 - 10\\
\operatorname{colog}\sin B &= 0.33464\\
\hline
\log c &= 1.05116\\
c &= 11.250.
\end{aligned}$$

(5) Check. By the double formula (Art. 66, (6)),

$$\frac{c}{a-b}=\frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)}.$$

$\log c = 1.05116$	$\log(a-b) = 0.82117$
$\log\sin\frac{1}{2}(A-B) = 9.69480 - 10$	$\log\sin\frac{1}{2}(A+B) = 9.92481 - 10$
0.74596	0.74598

Compact Arrangement

	Solution.	Check.	
$a = 12.346$			
$b = 5.7213$		log 0.75749	
$a - b = 6.6247$	log 0.82117		log 0.82117
$a + b = 18.0673$	colog 8.74310		
$C = 65^\circ\ 30'\ 10''$		log sin 9.95903	
$\frac{1}{2}(A+B) = 57^\circ\ 14'\ 55''$	log tan 0.19162		log sin 9.92481
$\frac{1}{2}(A-B) = 29^\circ\ 41'\ 02''$	log tan 9.75589		colog sin 0.30520
$A = 86^\circ\ 55'\ 57''$			
$B = 27^\circ\ 33'\ 53''$		colog sin 0.33464	
$c = 11.250$		1.05116	1.05118

Exercise 36

1. Given $a = 486$, $b = 347$, $C = 51^\circ\ 36'$.
   Find $A = 83^\circ\ 15'$, $B = 45^\circ\ 09'$, $c = 383.5$.

2. Given $a = 364$, $b = 640$, $C = 53^\circ\ 14'$.
   Find $A = 34^\circ\ 38'$, $B = 92^\circ\ 08'$, $c = 513$.

3. Given $a = 875$, $b = 567$, $C = 34^\circ\ 52'$. Find A, B, c.

4. Given $a = 233.4$, $b = 557.2$, $C = 18^\circ\ 23.0'$.
   Find $A = 12^\circ\ 22.0'$, $B = 149^\circ\ 15.0'$, $c = 343.7$.

5. Given $b = 145.9$, $c = 39.90$, $A = 92^\circ\ 11.3'$.
   Find $B = 72^\circ\ 40.7'$, $C = 15^\circ\ 08.0'$, $a = 152.7$.

6. Given $c = 453.9$, $a = 478.1$, $B = 35^\circ\ 37.9'$. Find C, A, b.

7. Given $a = 51.269$, $b = 14.687$, $C = 62^\circ\ 09'\ 24''$.
   Find $A = 101^\circ\ 32'\ 32''$, $B = 16^\circ\ 18'\ 04''$, $c = 46.269$.

8. Given $b = 467.92$, $c = 612.34$, $A = 45^\circ\ 29'\ 16''$.
   Find $B = 49^\circ\ 34'\ 05''$, $c = 84^\circ\ 56'\ 39''$, $a = 438.36$.

9. Given $c = 345.67$, $a = 654.32$, $B = 67° 45' 45''$. Find C, A, b.

10. Given $a = 447.45$, $b = 216.45$, $C = 116° 30' 20''$. Find the area.
Ans. $T = 43336$.

11. Show that when the included angle is a right angle, the law of tangents gives

$$\tan \tfrac{1}{2}(A - B) = \frac{a-b}{a+b}.$$

73. Case IV. Given three sides, *a*, *b*, *c*.

Each of the angles A, B and C is found by applying one of the formulas (2) or (4), Art. 68, for the tangent, sine or cosine of half the respective angle, but for the reasons stated in Art. 68 the tangent formulas are generally to be preferred.

To check, apply the relation $A + B + C = 180°$.

Example 1.

Given

$a = 12.653$,
$b = 17.213$,
$c = 23.106$.

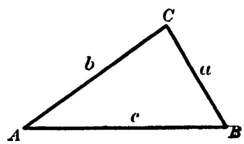


Fig. 80.

Required

$A = 32° 36' 22''$,
$B = 47° 08' 42''$,
$C = 100° 14' 56''$.

Solution. By formula (2), Art. 68,

$$\tan\frac{A}{2} = \frac{k}{s-a}, \quad \tan\frac{B}{2} = \frac{k}{s-b}, \quad \tan\frac{C}{2} = \frac{k}{s-c},$$

where

$$s = \frac{a+b+c}{2}, \quad k = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

$s = 26.486$		$\operatorname{colog} s =$	$8.57698 - 10$
$s - a = 13.833$		$\log(s-a) =$	1.14092
$s - b = 9.273$		$\log(s-b) =$	0.96722
$s - c = 3.380$		$\log(s-c) =$	0.52892
		$\log k^2 =$	1.21404
		$\log k =$	0.60702

$\log k = 0.60702$	$\log k = 0.60702$	$\log k = 0.60702$
$\log(s-a) = 1.14092$	$\log(s-b) = 0.96722$	$\log(s-c) = 0.52892$
$\log\tan\frac{1}{2}A = 9.46610$	$\log\tan\frac{1}{2}B = 9.63980$	$\log\tan\frac{1}{2}C = 0.07810$
$\frac{1}{2}A = 16° 18' 11''$	$\frac{1}{2}B = 23° 34' 21''$	$\frac{1}{2}C = 50° 07' 28''$
$A = 32° 36' 22''$,	$B = 47° 08' 42''$,	$C = 100° 14' 56''$.

Check. $A + B + C = 32° 36' 22'' + 47° 08' 42'' + 100° 14' 56'' = 180°$.

Compact Arrangement

	Solution.	Check.
$a = 12.653$		
$b = 17.213$		
$c = 23.106$		
$s = 26.486$	colog 8.57698	
$s - a = 13.833$	log 1.14092	
$s - b = 9.273$	log 0.96722	
$s - c = 3.380$	log 0.52892	
k^2	log 1.21404	
k	log 0.60702	
$\frac{1}{2}A = 16° 18' 11''$	log tan 9.46610	$A = 32° 36' 22''$
$\frac{1}{2}B = 23° 34' 21''$	log tan 9.63980	$B = 47° 08' 42''$
$\frac{1}{2}C = 50° 07' 28''$	log tan 0.07810	$C = 100° 14' 56''$
		$180° 00' 00''$

EXAMPLE 2. Find the area of the triangle and the radii of the inscribed and escribed circles for the triangle in Example 1.

Solution. $a = 12.653$, $b = 17.213$, $c = 23.106$.

The area of a triangle in terms of the sides is given by (4), Art. 67.

$$T = \sqrt{s(s-a)(s-b)(s-c)}.$$

The formulas for the radii of the inscribed and escribed circles are given in Problem 7, Exercise 33.

$$k = \frac{T}{s}, \quad k_a = \frac{T}{s-a}, \quad k_b = \frac{T}{s-b}, \quad k_c = \frac{T}{s-c}.$$

Using the results of Example 1, we have

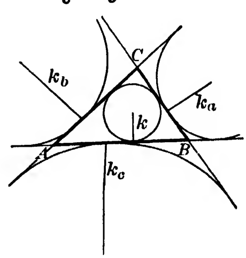


Fig. 81.

$$\begin{aligned}
\log s &= 1.42302 \\
\log (s-a) &= 1.14092 \\
\log (s-b) &= 0.96722 \\
\log (s-c) &= 0.52892 \\
\hline
\log T^2 &= 4.06008 \\
\log T &= 2.03004 & T &= 107.16, \\
\log k &= 0.60702 & k &= 4.046, \\
\log k_a &= 0.88912 & k_a &= 7.747, \\
\log k_b &= 1.06282 & k_b &= 11.556, \\
\log k_c &= 1.50112 & k_c &= 31.704.
\end{aligned}$$

Check. A convenient check is obtained by using the relation in Problem 8, Exercise 33,

$$\frac{1}{k} = \frac{1}{k_a} + \frac{1}{k_b} + \frac{1}{k_c}.$$

$$\log \frac{1}{k} = \operatorname{colog} k = 9.39298 - 10 \qquad \frac{1}{k} = 0.24716$$

$$\log \frac{1}{k_a} = \operatorname{colog} k_a = 9.11088 - 10 \qquad \frac{1}{k_a} = 0.12909$$

$$\log \frac{1}{k_b} = \operatorname{colog} k_b = 8.93718 - 10 \qquad \frac{1}{k_b} = 0.08653$$

$$\log \frac{1}{k_c} = \operatorname{colog} k_c = 8.49888 - 10 \qquad \frac{1}{k_c} = 0.03154$$

$$\frac{1}{k_a} + \frac{1}{k_b} + \frac{1}{k_c} = 0.24716 = \frac{1}{k}.$$

Exercise 37

1. Given $a = 286$, $b = 321$, $c = 463$.
   Find $A = 37° 34'$, $B = 43° 11'$, $C = 99° 15'$.
2. Given $a = 3.21$, $b = 3.61$, $c = 4.02$.
   Find $A = 49° 24'$, $B = 58° 38'$, $C = 71° 58'$.
3. Given $a = 74.6$, $b = 81.9$, $c = 90.0$. Find the angles.
4. Given $a = 354.4$, $b = 277.9$, $c = 401.3$.
   Find $A = 59° 39.5'$, $B = 42° 35.3'$, $C = 77° 45.2'$.
5. Given $a = 1.961$, $b = 2.641$, $c = 1.354$.
   Find $A = 46° 03.7'$, $B = 104° 07.6'$, $C = 29° 48.8'$.
6. Given $a = 87.06$, $b = 9.16$, $c = 79.02$. Find A, B, C.
7. Given $a = 3359.4$, $b = 4216.3$, $c = 4098.7$.
   Find $A = 47° 38' 00''$, $B = 68° 01' 06''$, $C = 64° 20' 54''$.
8. Given $a = 33.112$, $b = 44.224$, $c = 55.336$.
   Find $A = 30° 45' 14''$, $B = 53° 03' 08''$, $C = 90° 11' 38''$.
9. Given $a = 14.493$, $b = 55.436$, $c = 66.913$. Find the angles.
10. Given $a = 46.78$, $b = 35.90$, $c = 77.00$. Find the area.
    Ans. $T = 573.91$.
11. Find the area in Problem 2. Check.

12. Find the radii of the inscribed circle, of the escribed circles and the circumcircle of the triangle in problem 10.

Check by using the relation $k_a + k_b + k_c - k = 2\ D = 4\ R$, Problem 9, Exercise 33.

13. $a = 4$, $b = 5$, $c = 6$. Find the angles by applying the law of cosines directly, that is,

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc},\ \cos B = \text{etc.}$$

14. The sides of an inscribed quadrilateral taken in order are

$$a = 56,\ b = 33,\ c = 16,\ d = 63.$$

Required the area Q of the quadrilateral and the angles A, B, C, D, the notation being that employed in Fig. 71, Problem 10, Exercise 33.

Ans. $Q = 1428$, $A = 44°\ 46'$, $B = 90°$, $C = 135°\ 14'$, $D = 90°$.

74. Practical Applications. The solution of oblique triangles finds numerous applications in the various arts and sciences. Chief among these are surveying, engineering, physics, astronomy and navigation. The simpler applications which involve the solution of a single triangle need no explanation, since they may be immediately referred to some one of the four cases treated in the preceding sections. Many of the practical applications, however, involve several triangles which must be successively solved in whole or in part before the required distance or angle can be ascertained. Sometimes the intermediate triangles to be solved are not apparent from the figure, but must be sought by some auxiliary geometrical construction. Again, it may happen that no single triangle exists containing the requisite number of parts; in such cases the solution is effected by solving the equations which arise by applying the formulas of Chapter VII so as to involve the unknown parts. We shall illustrate each of these cases by an example.

(*a*) *System of triangles.* EXAMPLE 1. Given one side c of a quadilateral $ABC'C$ (Fig. 82) and two angles A and B adjacent to this side, also the angles α, β which the diagonals d_1, d_2, drawn from A and B respectively, make with the given side, to determine the side x opposite the given side c. This problem is sometimes known as ***Hansen's problem.***

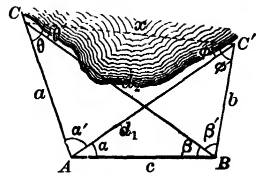


Fig. 82.

Analysis. Let $AC = a$, $BC' = b$, angle $CAC' = \alpha'$, angle $CBC' = \beta'$, angle $ACC' = \theta$, angle $BCC' = \theta'$, angle $AC'C = \phi$, angle $BC'C = \phi'$.

(1) In the triangle ABC, one side c and two adjacent angles A, β are known, hence d_2 may be found.

(2) In the triangle ABC', one side c and two adjacent angles α, B are known, hence b may be found.

(3) Then, in the triangle CBC', two sides d_2, b and the included angle $\beta' = B - \beta$ * are known, hence x may be found.

(4) Check. Compute x again, using the triangle CAC', having previously found a and d_1 from the triangles ABC and ABC' respectively.

Illustration. In order to determine the length CC' (Fig. 82) of a trestle to be built across the end of a lagoon, a distance AB, 500 ft. long, was measured off, and the following angles were measured with a transit:

$$CAB = 105^\circ\, 30' = A,\quad C'BA = 95^\circ\, 50' = B,$$
$$C'AB = 35^\circ\, 17' = \alpha,\quad CBA = 47^\circ\, 32' = \beta.$$

Required the distance $CC' = x$.

Solution. (1) Triangle ABC. By the law of sines

$$d_2 = c\frac{\sin A}{\sin ACB},\quad \text{and}\quad a = c\frac{\sin \beta}{\sin ACB}.$$

$$ACB = 180^\circ - (A + \beta) = 26^\circ\, 58'.$$

$\log c$	$= 2.69897$		$= 2.69897$
$\log \sin A$	$= 9.98391$†	$\log \sin \beta$	$= 9.86786$
colog $\sin ACB$	$= 0.34345$		$= 0.34345$
$\log d_2$	$= 3.02633$	$\log a$	$= 2.91028$
d_2	$= 1062.5.$	a	$= 813.36.$

* It is assumed that the points A, B, C, C' are in the same plane, otherwise the angles CAC' and CBC' must be given in addition to the data of the problem.

† $9.98391 - 10$. In this and the following problems the -10's are omitted when this may be done without danger of confusion. This is a common practice among computers.

(2) Triangle ABC'. By the law of sines

$$b = c\,\frac{\sin\alpha}{\sin AC'B}, \quad \text{and} \quad d_1 = c\,\frac{\sin B}{\sin AC'B}.$$

$$AC'B = 180^\circ - (B+\alpha) = 48^\circ\,53'.$$

$$\begin{aligned} \log c &= 2.69897 \\ \log\sin\alpha &= 9.76164 \\ \operatorname{colog}\sin AC'B &= \underline{0.12299} \\ \log b &= 2.58360 \\ b &= 383.35. \end{aligned} \qquad \begin{aligned} &= 2.69897 \\ \log\sin B &= 9.99775 \\ &= \underline{0.12299} \\ \log d_1 &= 2.81971 \\ d_1 &= 660.25. \end{aligned}$$

(3) Triangle CBC'. Triangle CAC'.

By the law of tangents

$$\tan\frac{\phi'-\theta'}{2} = \frac{d_2-b}{d_2+b}\tan\frac{\phi'+\theta'}{2}, \qquad \tan\frac{\phi-\theta}{2} = \frac{a-d_1}{a+d_1}\tan\frac{\phi+\theta}{2}.$$

$\frac{1}{2}(\phi'+\theta') = \frac{1}{2}(180^\circ-\beta) = 48^\circ\,18'$. $\frac{1}{2}(\phi+\theta) = \frac{1}{2}(180^\circ-\alpha') = 54^\circ 53.5'$.

$d_2-b=679.15$, $d_2+b=1445.85$, $a-d_1=153.11$, $a+d_1=1473.61$,

$$\begin{aligned} \log(d_2-b) &= 2.83196 \\ \operatorname{colog}(d_2+b) &= 6.83988 \\ \log\tan\tfrac{1}{2}(\phi'+\theta') &= \underline{0.34836} \\ \log\tan\tfrac{1}{2}(\phi'-\theta') &= 0.02020 \\ \tfrac{1}{2}(\phi'-\theta') &= 46^\circ\,19'\,55''. \end{aligned} \qquad \begin{aligned} \log(a-d_1) &= 2.18501 \\ \operatorname{colog}(a+d_1) &= 6.83162 \\ \log\tan\tfrac{1}{2}(\phi+\theta) &= \underline{0.15302} \\ \log\tan\tfrac{1}{2}(\phi-\theta) &= 9.16965 \\ \tfrac{1}{2}(\phi-\theta) &= 8^\circ\,24'\,25''. \end{aligned}$$

By the double formula involving the sines,

$$x = (d_2-b)\,\frac{\sin\frac{1}{2}(\phi'+\theta')}{\sin\frac{1}{2}(\phi'-\theta')}, \qquad x = (a-d_1)\,\frac{\sin\frac{1}{2}(\phi+\theta)}{\sin\frac{1}{2}(\phi-\theta)}.$$

$$\begin{aligned} \log(d_2-b) &= 2.83196 \\ \log\sin\tfrac{1}{2}(\phi'+\theta') &= 9.96022 \\ \operatorname{colog}\sin\tfrac{1}{2}(\phi'-\theta') &= \underline{0.14065} \\ \log x &= 2.93283 \\ x &= 856.7, \end{aligned} \qquad \begin{aligned} \log(a-d_1) &= 2.18501 \\ \log\sin\tfrac{1}{2}(\phi+\theta) &= 9.91278 \\ \operatorname{colog}\sin\tfrac{1}{2}(\phi-\theta) &= \underline{0.83505} \\ \log x &= 2.93284 \\ x &= 856.7 \text{ (check).} \end{aligned}$$

(*b*) *Auxiliary geometrical constructions.* EXAMPLE 2. In Example 1 it was shown how the distance CC' may be found without leaving the line AB. Another important problem is to determine one's position from the angles which the sides of a known triangle subtend from that position. This is known among surveyors as the three-

point problem, sometimes as *Pothenot's problem.** It may be stated thus: Given three points A, B, C whose mutual distances are known, to find the distance of a fourth point P from either of the points A, B, C, having given the angles, which the sides AB, BC, CA subtend at P.

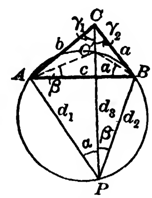


Fig. 83.

Analysis. Let A, B, C be the three points whose mutual distances are, —

$$AB = c,\ BC = a,\ CA = b,$$

and let P represent the fourth point at which AC and BC subtend the angles α and β respectively. It is required to find the distances d_1, d_2, d_3 of P from A, B and C respectively.

Circumscribe a circle about the triangle ABP, and let C' be the point at which this circle cuts PC or PC produced. Draw AC' and BC'. Then angle BAC' = angle $BPC = \beta$, being inscribed angles subtended by the same arc, and likewise angle ABC' = angle APC $= \alpha$.

(1) In the triangle ABC', one side c and the two adjacent angles α, β are known, hence AC' can be found.

(2) In the triangle ABC, the sides are known, hence A, the angle opposite the side a, can be found.

(3) In the triangle ACC', two sides b and AC' and the included angle $CAC' = A - \beta$ are known, hence angle $ACC' = \gamma_1$ can be found.

(4) In the triangle APC, one side b and two angles α, γ_1 are known, hence the sides d_1, d_3 can be found.

(5) Similarly, d_2 and d_3 can be found from the triangle BPC, having previously computed the triangle BCC'.

(6) Check. Compare the value of d_3 found in (4) with the value of d_3 in (5).

Illustration. A, B, C (Fig. 83) are three hostile forts whose mutual distances are known to be $AB = 4$ miles, $BC = 2$ miles, $CA = 3$ miles. From a battery planted at P, AC subtends an angle $34° 30'$ and BC an angle of $23° 45'$. Find the distance of the battery from each of the forts.

* Pothenot's problem and Hansen's problem are named after the men who were supposed to have first formulated and solved these problems. It is now known that both of these problems were previously solved by Snellius, the first in 1617, the other in 1627. Hence, if any one's name is to be associated with these problems hereafter it ought to be that of Snellius.

Solution. With the notation indicated in the figure, we have given

$$a = 2, \quad b = 3, \quad c = 4, \quad \alpha = 34^\circ\, 30', \quad \beta = 23^\circ\, 45';$$

to find d_1, d_2, d_3.

(1) Triangle ABC'. By the law of sines

$$AC' = c\frac{\sin\alpha}{\sin AC'B} = c\frac{\sin\alpha}{\sin(\alpha+\beta)}, \qquad BC' = c\frac{\sin\beta}{\sin(\alpha+\beta)}.$$

$$\alpha + \beta = 34^\circ\, 30' + 23^\circ\, 45' = 58^\circ\, 15'.$$

$$\begin{aligned} \log c &= 0.60206 & &= 0.60206 \\ \log\sin\alpha &= 9.75313 & \log\sin\beta &= 9.60503 \\ \operatorname{colog}\sin(\alpha+\beta) &= \underline{0.07040} & &= \underline{0.07040} \\ \log AC' &= 0.42559 & \log BC' &= 0.27749 \\ AC' &= 2.6644. & BC' &= 1.8945. \end{aligned}$$

(2) Triangle ABC. By the law of cosines

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc} \qquad \cos B = \frac{c^2 + a^2 - b^2}{2\,ca}$$

$$= \frac{3^2 + 4^2 - 2^2}{2\cdot 3\cdot 4} = 0.8750 \qquad = \frac{4^2 + 2^2 - 3^2}{2\cdot 4\cdot 2} = 0.6875$$

$$A = 28^\circ\, 57'. \qquad B = 46^\circ\, 34'.$$

(3) Triangle ACC'. Triangle BCC'.

By the law of tangents

$$\tan\frac{AC'C - \gamma_1}{2} = \frac{b - AC'}{b + AC'}\tan\frac{ACC' + \gamma_1}{2}, \qquad \tan\frac{BC'C - \gamma_2}{2} = \frac{a - BC'}{a + BC'}\tan\frac{BC'C + \gamma_2}{2}.$$

$$CAC' = A - \beta = 5^\circ\, 12', \qquad CBC' = B - \alpha = 12^\circ\, 04'.$$

$$\frac{AC'C + \gamma_2}{2} = \frac{180^\circ - 5^\circ\, 12'}{2} = 87^\circ\, 24', \quad \frac{BC'C + \gamma_2}{2} = \frac{180^\circ - 12^\circ\, 04'}{2} = 83^\circ\, 58',$$

$$\begin{aligned} b - AC' &= 0.3356, & a - BC' &= 0.1055, \\ b + AC' &= 5.6644. & a + BC' &= 3.8945. \\ \log(b - AC') &= 9.52582 & \log(a - BC') &= 9.02325 \\ \operatorname{colog}(b + AC') &= 9.24685 & \operatorname{colog}(a + BC') &= 9.40954 \\ \log\tan\frac{AC'C + \gamma_1}{2} &= \underline{1.34285} & \log\tan\frac{BC'C + \gamma_2}{2} &= \underline{0.97596} \end{aligned}$$

$$\log\tan\frac{AC'C-\gamma_1}{2} = 0.11552. \qquad \log\tan\frac{BC'C-\gamma_2}{2} = 9.40875$$

$$\frac{AC'C-\gamma_1}{2} = 55^\circ\ 32' \qquad \frac{BC'C-\gamma_2}{2} = 14^\circ\ 23'$$

$$\gamma_1 = \frac{AC'C+\gamma_1}{2} - \frac{AC'C-\gamma_1}{2} \qquad \gamma_2 = \frac{BC'C+\gamma_2}{2} - \frac{BC'C-\gamma_2}{2}$$

$$= 34^\circ\ 52'. \qquad = 69^\circ\ 35'.$$

(4) Triangle APC. Triangle BPC.

By the law of sines

$$d_1 = b\frac{\sin\gamma_1}{\sin\alpha}, \qquad d_2 = a\frac{\sin\gamma_2}{\sin\beta},$$

$$d_3 = b\,\frac{\sin CAP}{\sin\alpha} = b\,\frac{\sin(\alpha+\gamma_1)}{\sin\alpha} = a\,\frac{\sin(\beta+\gamma_2)}{\sin\beta}.$$

$$\alpha+\gamma_1 = 34^\circ 30' + 34^\circ\ 52' = 69^\circ\ 22',$$
$$\beta+\gamma_2 = 23^\circ 45' + 69^\circ 35' = 93^\circ\ 20'.$$

$$\begin{aligned} \log b &= 0.47712 \\ \log\sin\gamma_1 &= 9.75714 \\ \operatorname{colog}\sin\alpha &= \underline{0.24687} \\ \log d_1 &= 0.48113 \\ d_1 &= 3.0277. \end{aligned} \qquad \begin{aligned} \log a &= 0.30103 \\ \log\sin\gamma_2 &= 9.97182 \\ \operatorname{colog}\sin\beta &= \underline{0.39497} \\ \log d_2 &= 0.66782 \\ d_2 &= 4.6539. \end{aligned}$$

$$\begin{aligned} \log b &= 0.47712 \\ \log\sin(\alpha+\gamma_1) &= 9.97121 \\ \operatorname{colog}\sin\alpha &= 0.24687 \\ \log d_3 &= \underline{0.69520} \\ d_3 &= 4.957. \end{aligned} \qquad \begin{aligned} \log a &= 0.30103 \\ \log\sin(\beta+\gamma_2) &= 9.99925 \\ \operatorname{colog}\sin\beta &= 0.39497 \\ \log d_3 &= \underline{0.69526} \\ d_3 &= 4.957 \text{ (check)}. \end{aligned}$$

(*c*) *Solution by solving a system of simultaneous equations.*

EXAMPLE 3. From a point O (Fig. 84), the segments AP, PQ, QB of a straight line subtend the angles α, γ, β respectively. The distances $AB = d$, and $PQ = c$ are known; required the distances $AP = x$ and $QB = y$.

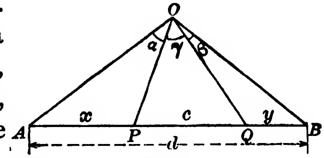


Fig. 84.

Solution. In this case no single triangle contains a sufficient number of known parts to afford a solution, but on applying the law of sines to the various triangles, we obtain the following equations,—

$$\frac{x}{\sin\alpha} = \frac{OP}{\sin A}, \quad (1) \qquad \frac{x+c}{\sin(\alpha+\gamma)} = \frac{OQ}{\sin A}, \quad (2)$$

$$\frac{y}{\sin\beta} = \frac{OQ}{\sin B}, \quad (3) \qquad \frac{y+c}{\sin(\beta+\gamma)} = \frac{OP}{\sin B}. \quad (4)$$

Dividing (1) by (2), and (3) by (4), we obtain

$$\frac{x}{x+c}\cdot\frac{\sin(\alpha+\gamma)}{\sin\alpha} = \frac{OP}{OQ}, \quad (5) \qquad \frac{y}{y+c}\cdot\frac{\sin(\beta+\gamma)}{\sin\beta} = \frac{OQ}{OP}. \quad (6)$$

Multiplying (5) by (6) gives, after a slight reduction,

$$\frac{xy}{(x+c)(y+c)} = \frac{\sin\alpha\sin\beta}{\sin(\alpha+\gamma)\sin(\beta+\gamma)}. \quad (7)$$

From the conditions of the problem we also have

$$x + y = d - c. \quad (8)$$

The simultaneous equations (7) and (8) contain but two unknowns, x and y, which may be found from them by the familiar methods of algebra.

Second solution. Call the common altitude of the triangles h (Fig. 84). The area of each triangle may be expressed in two ways, first as one-half the product of the base by the altitude, second as one-half the product of two sides into the sine of the angle included by these sides (Art. 67, (2)).

Accordingly we have

$$2\cdot\text{area } APO = xh = AO\cdot PO\cdot\sin\alpha \quad (9)$$

$$2\cdot\text{area } QPO = yh = QO\cdot BO\cdot\sin\beta \quad (10)$$

$$2\cdot\text{area } PQO = ch = PO\cdot QO\cdot\sin\gamma \quad (11)$$

$$2\cdot\text{area } ABO = dh = AO\cdot BO\cdot\sin(\alpha+\gamma+\beta) \quad (12)$$

Dividing the product of (9) and (10) by the product of (11) and (12), and canceling the factors which appear in both the numerator and the denominator, gives

$$\frac{xy}{cd} = \frac{\sin\alpha\sin\beta}{\sin\gamma\sin(\alpha+\beta+\gamma)}, \quad\text{or}\quad xy = cd\frac{\sin\alpha\sin\beta}{\sin\gamma\sin(\alpha+\beta+\gamma)}. \quad (13)$$

Equation (13), together with equation (8), is sufficient for the determination of x and y.

Illustration. An island PQ (Fig. 84) one mile wide lies in a direct line between two cities A and B on opposite shores of a river. From

CHAPTER IX

THE GENERAL ANGLE AND ITS MEASURES

81. General Definition of an Angle. In elementary geometry an angle is defined as the difference in direction, or the amount of opening, between two lines which meet or tend to meet in a point. Obviously, two lines differ most in direction, the opening between them is greatest, when they extend in opposite directions. According to this definition, no angle can be greater than a straight angle, and in fact every angle considered in plane geometry is less than two right angles.

For the purposes of trigonometry and higher mathematics in general it is convenient to think of angles as formed by revolving a line in a plane about a fixed point. The line in its first or initial position is called the *initial line*, in its final position it is called the *terminal line*, the fixed point is called the *vertex*. *The amount of rotation which brings the line from its initial position to its terminal position is called the angle between the two lines.*

With this definition, it is plain that an angle may have any magnitude whatever, for there is no limit to the amount of rotation which a line may undergo. The angle described by the big hand of a clock increases so long as the clock continues to run,— when it has run 15 minutes the angle described is a right angle; when it has run 30 minutes the angle described is a straight angle; after an hour the big hand has returned to its initial position, yet the angle which it has described is not zero but four right angles. As the hand continues to move, the angle between the hand and its initial position still increases. After two complete revolutions this angle is equal to eight right angles; after three complete revolutions, to twelve right angles, and so on.

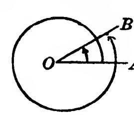


Fig. 118.

Before we can assign a magnitude to an angle in this wider sense, we must know more about it than merely the position of its sides. By the angle AOB (Fig. 118) may be meant an angle of $45°$, as indicated by the short arrow, or an angle of $405°$, as indicated by the long arrow, or

an angle of $45°$ increased by any number times $360°$, depending on how many complete revolutions OB has described in reaching its final position. The angles $45°$, $405°$, and all the other angles which AOB might represent, are said to be *coterminal*. In elementary geometry, and generally when nothing to the contrary is said or implied, in referring to an angle, the numerically smallest of all the coterminal angles is understood. This value is known as the *principal value* of the angle.

82. Positive and Negative Angles. Sometimes it is convenient to distinguish between the two directions in which rotation of a line about a point in a plane may take place. This is done by prefixing a plus sign to the resulting angle if it has been described by rotation in a direction contrary to that in which the hands of a clock move (counterclockwise), Fig. 119, and a minus sign if the rotation has been in the opposite direction (clockwise), Fig. 120. When the sign is considered, the initial line is always read first; thus, each of the angles in figures 119 and 120 is read "angle AOB."

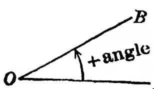


Fig. 119.

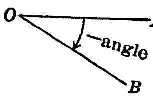


Fig. 120.

83. Complement and Supplement. If the algebraic sum of two angles is equal to a right angle, each angle is said to be the *complement* of the other; if their algebraic sum equals two right angles each is said to be the *supplement* of the other. That is to say,—

If $A + B = 90°$, A and B are complementary angles,

If $A + B = 180°$, A and B are supplementary angles.

A or B may have any magnitude, positive or negative. Thus, since

$95° + (-5°) = 90°$, $95°$ and $-5°$ are complementary angles,

and since

$-110° + 290° = 180°$, $-110°$ and $290°$ are supplementary angles.

84. Angles in the Four Quadrants. Let the vertex of the angle be taken as the origin of a system of rectangular coördinate axes, and the initial side of the angle for the positive direction of the x-axis. It follows that every angle less than $90°$, as angle AOP_1,

Fig. 121, has its terminal side in the first quadrant, and for this reason it is said to be an angle of the first quadrant. Every angle which is greater than a right angle but less than a straight angle, as angle AOP_2, is said to lie in, or to be an angle of, the second quadrant. Similarly, the angle AOP_3 is an angle of the third quadrant, angle AOP_4 an angle of the fourth quadrant, but an angle greater than four right angles and less than five right angles falls again in the first quadrant, and so on. Negative angles also lie in the quadrants determined by their terminal sides. Thus, while angle AOB (Fig. 119) lies in the first quadrant, the angle AOB (Fig. 120) lies in the fourth quadrant.

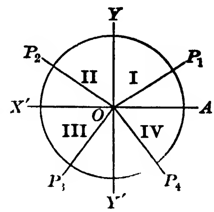


Fig. 121.

85. Sexagesimal Measure of Angles. It is familiar to all that different units of measures may be employed in the measurement of a given magnitude. Thus, any distance may be expressed in inches, feet, yards, rods and miles of the English system, or in millimeters, centimeters, meters and kilometers of the metric system. Values may be expressed in dollars and cents, in pounds and shillings, in francs and centimes, etc. Just so, angular magnitudes may be expressed in terms of different units of measure. In order to express a measure given in one set of units in terms of the units of another system, it is only necessary to know the relation between the units of the two systems.

The system of angular measure which we have used thus far is called the *sexagesimal system.** The total angular magnitude about

* From the Latin word *sexagesimus* (*sexa*, six + *decimus*, one-tenth), meaning one-sixtieth.

The sexagesimal scale was once applied to many measures. The sixtieth part of a unit of time, and length and weight was called its primate or prime; each prime was divided into sixty seconds, each second into sixty thirds. The divisor 60 has now disappeared among Western nations, except in measures of angles and of time. It is probable that the division of the angular space about a point into 360 degrees originated with the Babylonians, whose year consisted of 360 days. The Latin word for degree is *gradus*, the Greek, *bathmos*, each of which means step; that is, a degree originally meant the daily step of the sun eastward among the stars. The Chinese knew many centuries ago that the year consisted more nearly of $365\frac{1}{4}$ days, so they divided the total angular space about a point into $365\frac{1}{4}$ degrees. The sexagesimal system is unscientific and is doomed to be replaced sooner or later by a better system.

a point is divided into four equal parts and each part is called a right angle. The ninetieth part of a right angle is called a degree (°), the sixtieth part of a degree is called a minute (′), and the sixtieth part of a minute is called a second (″). The sexagesimal system of angular measures is the one most frequently used in every-day life.

86. Decimal Division of Degrees. Instead of subdividing into minutes and seconds, the degree is sometimes divided decimally into tenths, hundredths and thousandths. The decimal division of the degree has been used more or less ever since the invention of decimal fractions in the sixteenth century. Tables based on the decimal division of the degree have been published at various times.*

87. Centesimal Measure of Angles. Another system of angular measure, known as the *centesimal*, is obtained by dividing the right angle into 100 equal parts, each of which is called a grade (*g*), each grade into 100 minutes (\`), and each minute into 100 seconds (\`\`). An angle of 25 grades 16 minutes and 78 seconds would be written

$$25^g\ 16^{`}\ 78^{``}, \text{ or more simply, } 25.1678^g.$$

The centesimal system of angular measures was introduced as a part of the metric system by the French reformers† at the time of the great revolution. It possesses many advantages over the sexagesimal system, but owing to the fact that nearly all reference books and tables, all records of observation, the graduation of all astronomical instruments and of most engineering instruments, as well as the scales of geographical and nautical maps and charts, are based on the sexagesimal system, the progress of the introduction has been slow. But in spite of the many obstacles to be overcome the system is making steady gains in the countries where the metric system is used. The centesimal system is now used exclusively in the field surveys in France, Belgium, Hessia and Baden, and it is legally recognized in several other states. It is regularly taught in many European high schools and technical schools.

* The decimal system is now taught alongside the sexagesimal system in Harvard University and a number of other Eastern institutions of learning.

† The invention of this system and the first attempt to introduce it dates back to 1783 and is due to a German by the name of Johann Karl Schultze.

88. The Circular or Natural System of Angular Measures. In many practical investigations and in nearly all theoretical work, it is convenient to employ what is known as the circular or natural system of angular measure.

It is shown in geometry that in concentric circles the arcs a and a' (Fig. 122) intercepted by any angle α at the center are proportional to the radii r and r' of the circles, that is,

$$\frac{a}{r} = \frac{a'}{r'}, \tag{1}$$

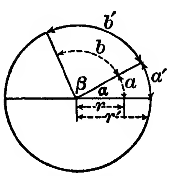


Fig. 122.

and again, that in the same circle two central angles α and β are to each other as their intercepted arcs a and b, that is,

$$\frac{\alpha}{\beta} = \frac{a}{b} = \frac{a}{r}\bigg/\frac{b}{r} \tag{2}$$

From (1) it follows that the ratio of the length of the arc to the radius of the circle is independent of the length of the radius, — in other words, that this ratio is constant so long as the angle is constant — and from (2) that this ratio varies as the angle, and may therefore be used as the measure of the angle. This ratio is known as the circular measure of the angle.

The circular measure of an angle is the ratio of the length of its intercepted arc, in a circle whose center is at the vertex of the angle, to the radius of the circle.

The unit of circular measure or *natural unit* is obtained by making a equal to r, that is, by taking the angle such that the intercepted arc equals the radius. This unit is called a radian.

A radian is an angle which, when placed with its vertex at the center of a circle, intercepts an arc equal in length to the radius of the circle.

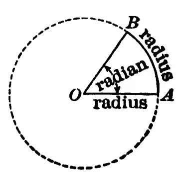


Fig. 123.

Thus, if the arc AB (Fig. 123) is equal in length to the radius OA, the angle AOB measures one radian.

One peculiarity of the circular system is that it has no subsidiary or derived units; that is, no other units which are multiples or sub-multiples of a radian. All angles large or small are expressed in terms of this single unit, the radian.

89. Comparison of Sexagesimal and Circular Measure. To find the relation between the two kinds of units, degrees and radians, it is best to compare the two measures for the entire angular space about a point, that is,. of four right angles. Expressed in circular measure, the measure of this angular space is equal to the circumference of a circle divided by the radius. Now the circumference of a circle is equal to the diameter, or twice the radius, multiplied by 3.14159+. Denoting this number by the Greek letter π,* we have for the circular measure of four right angles

$$\frac{\text{circumference}}{\text{radius}} = 2\,\pi.$$

Measured in degrees the same angular space is 360°, hence we obtain the fundamental relation

$$2\,\pi \text{ radians} = 360 \text{ degrees},$$

or $$\boldsymbol{\pi} \textbf{ radians} = \textbf{180 degrees.} \tag{1}$$

This relation enables us to reduce radians to degrees, and vice versa. Dividing both sides of (1) by the number π, we obtain

$$1 \text{ radian} = \frac{180}{\pi} \text{degrees},$$

hence, —

* The number π is a most marvelous number. It is incommensurable, that is, it cannot be exactly expressed by a fraction whose terms are whole numbers. Nor can it be found by taking the square root, cube root or higher root of some commensurable number. Neither is it the root of any algebraic equation. For this reason it is called a transcendental number.

However, the value of π may be computed to any desired degree of accuracy. Archimedes showed that it is less than $3\frac{1}{7}$ and greater than $3\frac{10}{71}$. The former value is still commonly employed in rough approximations.

The Hindus, as early as the sixth century, computed the value of π from the perimeter of a regular inscribed polygon of 384 sides and found $\pi = 3.1416$; the value now generally used where $3\frac{1}{7} = 3.1428$ is not sufficiently accurate.

In recent times, the value of π has been computed to 707 places of decimals. The first 32 places were computed by Ludolph van Ceulen, a Dutchman, who devoted a good portion of his life to this task. For this reason π is frequently referred to as Ludolph's number. Its first 30 places are

$$\pi = 3.141{,}592{,}653{,}589{,}793{,}238{,}462{,}643{,}383{,}279.$$

To reduce radians to degrees, multiply the number of radians by $\frac{180}{\pi}$ (equals 57.3 nearly*).

Again, dividing both sides of (1) by 180 and writing the second member first, we obtain

$$1 \text{ degree} = \frac{\pi}{180} \text{ radians};$$

hence,—

To reduce degrees to radians, multiply the number of degrees by $\frac{\pi}{180}$ (equals 0.0175 nearly).

EXAMPLE 1. How many degrees in $\frac{3\pi}{2}$ radians?

Solution. Since π radians $= 180°$, $\frac{\pi}{2}$ radians $= 90°$,

and $\frac{3\pi}{2}$ radians $= 270°$.

EXAMPLE 2. How many radians in an angle of 60°?

Solution. Since $180° = \pi$ radians, $60° = \frac{\pi}{3}$ radians.

It is best to retain the letter π in the answers unless there is some reason to the contrary. For instance, putting for π its value 3.1416 and dividing by 3, the answer to the second example might have been written $60° = 1.0472$ radians, but $\pi/3$ radians is preferable. Likewise the first example might have been stated thus: How many degrees in 4.7144 radians? but the statement as first given is preferable.

When a number represents the measure of an angle, and no unit is expressed, the natural unit is understood. Thus, when we speak of the angle $\pi/2$, we mean not $\pi/2$ degrees but $\pi/2$ radians. The angle π means π radians or 180°, the angle 2 means not 2° but 2 radians, or 114.5916°. We may look upon $\frac{1}{2}\pi$, π, 2π, $2n\pi$, when referring to angles, as abbreviations of 90°, 180°, 360° and n times 360° respectively.

* More accurately,

1 radian = 57.295,779,513 degrees,
1 degree = 0.017,453,290 radians,
1 minute = 0.000,290,888 radians,
1 second = 0.000,004,848 radians.

The value of the radian in degrees has been calculated to 43 places of decimals.

EXERCISE 44

(Use $3\frac{1}{7}$ for π and 57.3 for the value of 1 radian. n represents any positive integer.)

1. Express decimally the following angles:

$$45^\circ\,48'\,36'',\ 185^\circ\,59'\,15'',\ 35^\circ\,30'\,30'',\ 375^\circ\,00'\,47''.$$

Ans. 45.81°, 185.9875°, 35.5083°, 375.01305°.

2. Express in degrees, minutes and seconds, 16.35°, 153.156°, 67.003°, $\frac{5}{7}$ of a right angle.

Ans. $16^\circ\,21'$, $153^\circ\,09'\,21.6''$, $67^\circ\,00'\,10.8''$, $64^\circ\,17'\,08.57''$.

3. Express in radians, 90°, 45°, $22\frac{1}{2}^\circ$, 60°, 15°, 30°, 135°, 270°. -315°, -75°, 360°. *Ans.* $\frac{\pi}{2}$, $\frac{\pi}{4}$, $\frac{\pi}{8}$, $\frac{\pi}{3}$, $\frac{\pi}{12}$, etc.

4. Express in sexagesimal units the following angles:

$$\frac{\pi}{18},\ \frac{\pi}{10},\ \frac{\pi}{5},\ 2\tfrac{1}{2}\pi,\ \frac{2\pi}{3},\ 2\pi,\ 2n\pi,\ 2,\ \tfrac{1}{2},\ \frac{1}{\pi}.$$

Ans. 10°, 18°, . . . 114.6°, 28.65°, 18.23°.

5. Give a geometrical representation of each of the following angles: 340°, $\frac{5}{2}$ right angles, $-\frac{3}{4}\pi$, 725°, 2π, $2n\pi$, $(2n+1)\pi$.

6. State in which quadrant a line is after describing the following angles: 105°, -105°, $\frac{5\pi}{4}$, $-\frac{1}{2}\pi$, $\frac{2\pi}{3}$, 2, $+375^\circ$, $2n\pi - \frac{1}{4}\pi$.

Ans. 2d, 3d, . . . 1st, 4th.

7. Name three angles coterminal with an angle 30°.

8. Name two positive and two negative angles coterminal with -15°. *Ans.* 345°, 705°, -375°, -735°.

9. What is the smallest positive angle coterminal with 465°? With -465°. *Ans.* 105°, 255°.

10. Name the smallest negative angle coterminal with 735°? With -625°?

11. The principal value of an angle is $\frac{1}{4}\pi$; write in one formula all coterminal angles. *Ans.* $2n\pi + \frac{1}{4}\pi$.

12. If θ represents any angle less than 90°, all angles in the first quadrant can be expressed by $2n\pi + \theta$. Express similarly all angles in each of the other three quadrants.

Ans. II. $(2n+1)\pi - \theta$. III. $(2n+1)\pi + \theta$. IV. $2n\pi - \theta$.

13. What is the supplement of the angle $\frac{1}{2}\pi + \theta$? What is the complement? *Ans.* $\frac{1}{2}\pi - \theta$, $-\theta$.

14. What is the complement of $\frac{1}{4}\pi + \theta$? Of $\frac{\pi}{3} - \theta$?

Ans. $\frac{1}{4}\pi - \theta, \frac{\pi}{6} + \theta.$

15. Express both in radians and in degrees:
(*a*) Each of the angles of an isosceles right triangle.
(*b*) Each angle of a triangle whose angles are to each other as 1 : 2 : 3.

16. Find the number of radians in each of the angles of a regular pentagon, octagon, n-gon. *Ans.* $\frac{3\pi}{5}, \frac{3\pi}{4}, \frac{(n-2)\pi}{n}.$

17. Express decimally the following angles: $17^g\ 18^{`}\ 19^{``}$, $25^g\ 65^{`}\ 75^{``}$, $187^g\ 05^{`}\ 95^{``}$, $1^g\ 00^{`}\ 15^{``}$. *Ans.* 17.1819^g, 25.6575^g, etc.

18. Express $35^\circ\ 30'\ 30''$ and $35^g\ 30^{`}30^{``}$ each as a fraction of a right angle. *Ans.* 0.394537, 0.35303.

19. Express $35^\circ\ 30'\ 30''$ in centesimal units. *Ans.* $39^g\ 45^{`}\ 37^{``}$.

20. Prove the following rule: To convert grades into degrees, diminish the number of grades by one-tenth of itself.

21. Make a corresponding rule for converting degrees into grades.

22. Find two regular polygons such that the number of degrees in an angle of one is to the number of degrees in an angle of the other as the number of sides of the first is to the number of sides of the second. *Ans.* Triangle and hexagon.

23. Find three pairs of regular polygons such that the number of degrees in an angle of one is equal to the number of grades in an angle of the other.

24. If g, d and r represent respectively the number of grades, degrees and radians in any angle, prove that

$$\pi(g - d) = 20r.$$

90. Relation Between Angle, Arc and Radius. If an angle θ is subtended by an arc 20 ft. long and the radius of the circumference of which the arc forms a part is 10 ft., the number of radians in θ is 20 ft. ÷ 10 ft. or 2, and generally, if an angle θ is subtended by an arc s units long and the radius measures r of the same units, the radian measure of the angle is

$$\boldsymbol{\theta = \frac{s}{r}}. \tag{1}$$

Solving this equation for s and r respectively, we get

$$s = r\theta, \qquad (2) \qquad\qquad r = \frac{s}{\theta}. \qquad (3)$$

Equation (1) enables us to find the angle when we know the length of the arc and the radius, (2) gives us the length of the arc in terms of the radius and the angle, and (3) gives the radius when the length of the arc is known as well as the angle which it subtends.

These three equations enable us to solve a great variety of interesting problems.

EXAMPLE 1. The diameter of the earth subtends an angle of $17.5''$ at the center of the sun. Assuming the diameter of the earth to be 7917.6 miles, what is the distance of the earth from the sun?

Solution. We may consider the diameter of the earth approximately equal to an arc which subtends an angle of $17.5''$. Reducing $17.5''$ to radians we find

$$\theta = \frac{17.5}{60 \times 60} \times \frac{\pi}{180};$$

hence, applying equation (1) we have for the required distance in miles

$$r = \frac{s}{\theta} = \frac{7917.6 \times 60 \times 60 \times 180}{17.5 \times \pi} = 93{,}322{,}000 \text{ miles.}$$

(By the use of logarithms, putting $\pi = 3.1416$.)

EXAMPLE 2. A railroad train going due north strikes a curve of 3500 feet radius. The curve turns to the left and the train leaves the curve just 40 seconds after it entered the curve. In what direction does the train move after it leaves the curve, assuming that it is going at a uniform rate of 50 miles per hour.

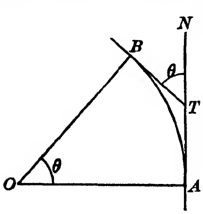


Fig. 124.

Solution. Let AN be the direction of the train on entering the curve and TB its direction when it leaves the curve at B.

It is required to find the angle NTB or θ.

Angle NTB = angle AOB (Why?), and by equation (1)

$$\theta = \text{angle } AOB \text{ (in radians)} = \frac{\text{arc } AB}{OA}.$$

OA is given and arc AB can be found, since we know the time which it takes the train to move over it at a given speed.

$$OA = \frac{3500}{5280} \text{ miles}, \qquad \text{arc } AB = \frac{50 \times 40}{60 \times 60} \text{ miles},$$

hence

$$\theta = \frac{50 \times 40}{60 \times 60} \div \frac{3500}{5280} = \frac{88}{105} \text{ radians},$$

$$= \frac{88}{105} \times \frac{180}{\pi} = \frac{88 \times 180 \times 7}{105 \times 22} = 48°.$$

The train leaves the curve in the direction N. 48° W.

90*a*. Area of a Circular Sector. The area of a circular sector is equal to the area of a triangle with a base equal in length to the arc of the sector and an altitude equal to the radius, hence the area A is

$$A = \frac{rs}{2},$$

which by (2), Article 90, becomes

$$A = \tfrac{1}{2} r^2 \theta,$$

where θ is expressed in radians. (1)

EXAMPLE 1. Find the area of a circular sector bounded by an arc 10 feet in length, the radius of the circle being 50 feet.

Solution. By (1), Article 90, the radian measure of the angle $\theta = \frac{10}{50}$, and $r = 50$, hence by (1)

$$A = \frac{1}{2} \cdot 50^2 \cdot \frac{10}{50} = 250 \text{ sq. ft.}$$

EXERCISE 45

(To obtain the answers as given, use $\pi = 3\frac{1}{7}$).

1. Express in radians and in degrees the angle subtended by an arc 50 ft. long on a circle whose radius is 200 ft.

Ans. $\frac{1}{4}$ radian $= 14.3°$.

2. What is the radius of a circle on which an arc 100 ft. long subtends an angle of 1°? *Ans.* $\frac{18{,}000}{\pi}$ ft.

3. How long is the arc subtended by a central angle of 75°, on a circumference whose radius is 100 ft.? *Ans.* 130.95 ft.

4. Find the length of an arc of 1° on the earth's surface ($r = 3960$ miles). *Ans.* $69\frac{1}{7}$ miles.

5. How long must a line be to subtend $1''$ at the distance of 1 mile? *Ans.* 0.3 inch (approximately).

6. A fly-wheel 10 ft. in diameter is revolving 200 revolutions per minute. Find the speed per second of a point on the rim.

Ans. $\frac{100\pi}{3}$ ft.

7. The diameter of the sun subtends an angle of $32'$ at the earth and the distance of the sun is about 93,000,000 miles. Required the diameter of the sun. *Ans.* 866,000 miles.

8. The earth moves around the sun once each year. Assuming its path to be a circle, find the velocity of the earth per second.

(Use the distance of the sun given in Problem 7.)

Ans. 18.5 miles.

9. The earth revolves on its axis once in 24 hours. Find its angular velocity per second, that is, the angle through which the earth turns per second, and hence the velocity in miles of a point on the equator. Use $r = 3960$ miles.

Ans. Angular velocity $15''$ per sec., linear velocity 0.288 mi. per sec.

10. The latitude of the city of Seattle is $47^\circ 40'$. Find the shortest distance from Seattle to the north pole. *Ans.* 2926 miles.

(Distances on the earth are of course measured along the arc of a great circle. Use $r = 3960$ mi.)

11. The diameter of a graduated circle is 12 inches and the graduations on its rim are $15'$. Find the approximate distance between two consecutive divisions on the rim. *Ans.* $\frac{1}{40}$ of an inch.

12. Two railroads meet at right angles at O. They are connected by a quadrant of a circle. The inner curve is 2000 ft. long. What is the distance of either point A or B from O?

Ans. $OA = \frac{4000}{\pi}$ ft.

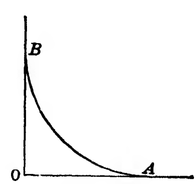


Fig. 125.

13. In Problem 12, if the rails are 4 ft. apart, how much longer will be the outer rail than the inner rail of the curve? *Ans.* 2π ft.

14. A circular sector, radius 10 inches, has an area of $26\frac{4}{21}$ sq. in. Find the angle of the sector. *Ans.* 30°.

15. A city lot has the shape of a circular sector, the curve bordering on the street. The straight sides of the lot are 100 ft. each and the angle between them is 60°. The lot was sold at $100 per foot frontage, what was the price per acre? *Ans.* $87,120.

91. Review.

1. (*a*) State and prove the law of sines. (*b*) State and prove the law of cosines. (*c*) Show that $2R = \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$, where R is the radius of the circumscribed circle. (*d*) Prove the projection theorem by means of the law of cosines.

2. (*a*) In any triangle show that $\sin(A + B) = \sin C$, $\cos(A + B) = -\cos C$, $\sin\frac{1}{2}(A + B) = \cos\frac{1}{2}C$, $\tan\frac{1}{2}(A + B) = \cot\frac{1}{2}C$. (*b*) Prove the double formula and the law of tangents. (*c*) Apply the law of tangents to the two legs a and b and the opposite acute angles of a right triangle and obtain $\tan\frac{1}{2}(A - B) = \dfrac{a - b}{a + b}$.

3. Express the area of a triangle: (*a*) In terms of two sides and the included angle. (*b*) In terms of the three sides. (*c*) Interpret geometrically the quantities $s - a, s - b, s - c$, and $k = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$.

4. Show how to solve a triangle when the three sides are given, — (*a*) Without logarithms. (*b*) With logarithms. (*c*) Write down the formulas needed in (*a*) and (*b*). (*d*) Prove the half angle formulas in terms of the sides.

5. Discuss the solution of each of the four cases of oblique triangles giving in each case the formulas necessary for the logarithmic solution and a formula for checking the answers.

6. Review the method of solving each of the four cases of oblique triangles by division into right triangles.

7. Given the three sides of a triangle as follows:

$$a = 301.9, \qquad b = 673.1, \qquad c = 422.8.$$

Compute one or more of each of the following sets of related quantities:

(*a*) Angles,	$A = 18° 12.4'$,	$B = 135° 50.8'$,	$C = 25° 56.9'$.
(*b*) Altitudes,	$h_a = 294.5$,	$h_b = 132.1$,	$h_c = 210.3$.
(*c*) Medians,	$m_a = 541.4$,	$m_b = 147.3$,	$m_c = 476.9$.
(*d*) Angle bisectors,	$b_A = 512.8$,	$b_B = 132.4$,	$b_C = 406.2$.
(*e*) Area and radii of inscribed and circumscribed circles,	$T = 44{,}458$,	$r = 63.61$	$R = 483.1$.
(*f*) Radii of escribed circles,	$r_a = 112.0$,	$r_b = 1723$,	$r_c = 161.0$.

(*g*) Check results by the graphic method.

8. (*a*) Give a general definition of an angle of any magnitude. (*b*) Define a negative angle. (*c*) Give a general definition of the complement and supplement of an angle. (*d*) What is meant by the principal value of a set of coterminal angles. (*e*) Give three positive and three negative angles coterminal with 30°.

9. (*a*) Explain each system of angular measures and define the unit in each. (*b*) State some of the advantages of each system. (*c*) State the relation between radians and degrees. (*d*) Express the following relations in radian measure, $\sin(180° - \theta) = \sin\theta$, $\tan(90° - \theta) = \cot\theta$, $\cos 180° = -1$.

10. Prove the following relations, where a denotes the side of any regular polygon, p the perimeter, n the number of sides, r the radius of the inscribed circle, R the radius of the circumscribed circle, and A the area of the polygon.

(*a*) $a = 2R\sin\frac{\pi}{n} = 2r\tan\frac{\pi}{n}$.

(*b*) $p = 2nR\sin\frac{\pi}{n} = 2nr\tan\frac{\pi}{n}$.

(*c*) $A = nR^2\sin\frac{\pi}{n}\cos\frac{\pi}{n} = \frac{nar}{2} = nr^2\tan\frac{\pi}{n}$.

CHAPTER X

FUNCTIONS OF ANY ANGLE

In this chapter a right angle will be denoted by R, rather than by $90°$ or $\pi/2$. The advantage of this notation is that it frees our results from any particular system of measure, that is, our results hold equally well whether the angles are expressed in degrees, grades or radians. Whenever R is expressed in a given unit it is understood that the other angles are expressed in the same unit. Thus, if we write $\sin (90° - \theta)$ it is understood that θ is expressed in degrees, but if we write $\sin (\pi/2 - \theta)$, θ is to be expressed in radians, while the expression $\sin (R - \theta)$ does not specify the unit in which the angles are measured.

92. Definition of the Trigonometric Functions of Any Angle. The functions of any angle, positive or negative, are defined in exactly the same way as were the functions of an angle less than $180°$. Let θ be any angle. Take the vertex O of the angle for an

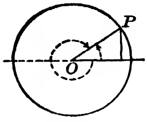


Fig. 126.

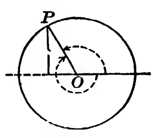


Fig. 127.

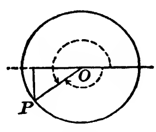


Fig. 128.

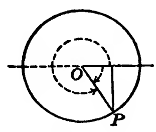


Fig. 129.

origin and the initial side of the angle for the positive direction of the x-axis. Let P represent any point (not the origin) on the terminal side of the angle. Let x and y denote the rectangular coördinates of P, and r its distance from the origin. Then whether θ is positive or negative, and whether P falls in the first, second, third or fourth quadrant, we have in every case,

$$\sin \theta = \frac{y}{r} = \frac{\text{ordinate}}{\text{distance}}, \qquad \csc \theta = \frac{1}{\sin \theta},$$

$$\cos\theta = \frac{x}{r} = \frac{\text{abscissa}}{\text{distance}}, \qquad \sec\theta = \frac{1}{\cos\theta},$$

$$\tan\theta = \frac{y}{x} = \frac{\text{ordinate}}{\text{abscissa}}, \qquad \cot\theta = \frac{1}{\tan\theta}.$$

93. Signs of the Functions in Each of the Quadrants.

(*a*) Since the sine is the ratio of the ordinate of P to its distance from O, and the distance is always positive, the sign of the sine is the same as the sign of the ordinate, which is + in the first and second, − in the third and fourth quadrants.

(*b*) Since the cosine is the ratio of the abscissa of P to its distance from O, and the distance is always positive, the sign of the cosine is the same as the sign of the abscissa, which is + in the first and fourth, − in the second and third quadrants.

(*c*) Since the tangent is the ratio of the ordinate of P to the abscissa of P, the sign of the tangent is + when the ordinate and abscissa have like signs, that is, in the first and third quadrants, and − when they have unlike signs, that is, in the second and fourth quadrants.

(*d*) Any number and its reciprocal have like signs, hence the signs of the cosecant, secant and cotangent are the same as the signs of the sine, cosine and tangent respectively.

The student must make himself perfectly familiar with the signs of the functions in the various quadrants. The following figure will prove an aid to his memory.

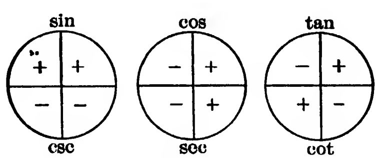


Fig. 130.

94. Periodicity of the Trigonometric Functions. If the terminal side of an angle is revolved in the plane of the angle through 4 R, or any number of times 4 R, it will return to the position from which it started, and this is true whether the revolution is in the positive (counterclockwise) or negative (clockwise) direction. It follows that the trigonometric functions of any angle remain unchanged

when the angle is increased or diminished by $4R$ or by any number of times $4R$. That is,

$$\sin(\theta \pm 4R) = \sin\theta,$$

and similarly for each of the other functions. In general,

$$\left.\begin{aligned} \sin(\theta \pm 4nR) &= \sin\theta, \\ \cos(\theta \pm 4nR) &= \cos\theta, \\ \tan(\theta \pm 4nR) &= \tan\theta, \text{etc.,} \end{aligned}\right\} \qquad (1)$$

where n is any positive or negative integer.

Thus,

$$\sin 375^\circ = \sin(375^\circ - 360^\circ) = \sin 15^\circ,$$
$$\sin(-15^\circ) = \sin(-15^\circ + 360^\circ) = \sin 345^\circ,$$
$$\sin\frac{9\pi}{4} = \sin\left(\frac{9\pi}{4} - 2\pi\right) = \sin\frac{\pi}{4},$$
$$\sin\frac{-7\pi}{6} = \sin\left(\frac{-7\pi}{6} + 2\pi\right) = \sin\frac{5\pi}{6},$$

and similarly for any other function.

It should be observed that by means of formula (1) the function of any negative angle can be replaced by the same function of some positive angle.

Since the trigonometric functions remain unchanged when the angle is increased or diminished by $4R$, they are called *periodic functions* with the *period* $4R$. It will be shown presently that the tangent and cotangent have the smaller period $2R$. There are other periodic functions besides the trigonometric functions.

95. Changes in the Values of the Functions while the Angle Changes from 0 to 4 R. Let a point P start from a position A and move in the positive direction along the circumference of a circle whose radius OA equals unity. Join P to the center O of the circle and let x and y represent the coördinates of P in its various positions with reference to O as origin and OA as the positive x-axis. Let us consider the changes in the values of the various functions of the angle $AOP = \theta$, as the point P moves along the circumference of the circle.

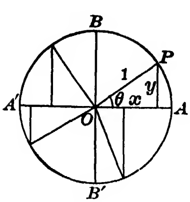


Fig. 131.

By definition $\sin\theta = \frac{y}{OP}$, $\cos\theta = \frac{x}{OP}$, $\tan\theta = \frac{y}{x}$. But OP was taken equal to 1, hence $\sin\theta$ is represented by the ordinate y, $\cos\theta$ by the abscissa x, and $\tan\theta$ by the ratio of the two.

First quadrant. While P moves from A to B, that is, while θ changes from 0 to R,
y is positive and changes from 0 to 1,
hence $\sin\theta$ is positive and changes from 0 to 1;
x is positive and changes from 1 to 0,
hence $\cos\theta$ is positive and changes from 1 to 0;
y/x is positive and changes from 0 to ∞,
hence $\tan\theta$ is positive and changes from 0 to ∞.

Second quadrant. While P moves from B to A', that is while θ changes from R to $2R$,
y is positive and changes from 1 to 0,
hence $\sin\theta$ is positive and changes from 1 to 0;
x is negative and changes from -0 to -1,
hence $\cos\theta$ is negative and changes from -0 to -1;
y/x is negative and changes from $-\infty$ to -0,
hence $\tan\theta$ is negative and changes from $-\infty$ to -0.

Third quadrant. While P moves from A' to B', that is, while θ changes from $2R$ to $3R$,
y is negative and changes from -0 to -1,
hence $\sin\theta$ is negative and changes from -0 to -1;
x is negative and changes from -1 to -0,
hence $\cos\theta$ is negative and changes from -1 to -0;
y/x is positive and changes from 0 to ∞,
hence $\tan\theta$ is positive and changes from 0 to ∞.

Fourth quadrant. While P moves from B' to A, that is, while θ changes from $3R$ to $4R$,
y is negative and changes from -1 to -0,
hence $\sin\theta$ is negative and changes from -1 to -0;
x is positive and changes from 0 to 1,
hence $\cos\theta$ is positive and changes from 0 to 1;
y/x is negative and changes from $-\infty$ to -0,
hence $\tan\theta$ is negative and changes from $-\infty$ to -0.

Cosecant, secant and cotangent. Since these functions are the reciprocals of the sine, cosine and tangent respectively, their variations can be immediately written down from the variations of the latter. Remember, —

(*a*) That the reciprocal of a number less than 1 is some number greater than 1, and vice versa.

(*b*) That the reciprocal of 0 is ∞, and vice versa.

(*c*) That reciprocals have like signs.

96. Changes in the Value of the Tangent of an Angle as the Angle Changes From 0 to 4 *R*. Some students find it difficult to follow the changes in the tangent from the ratio y/x when x and y both change. The following discussion is free from this difficulty.

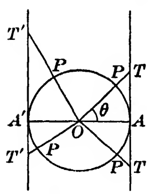


Fig. 132.

Let P (Fig. 132) move as in Fig. 131, but instead of the coördinates of the point P let us consider the coördinates of the point T or T' in which OP produced meets one of the tangents to the circle at A and A'.

First quadrant. While θ changes from 0 to R, AT changes from 0 to ∞, therefore $\tan\theta = \dfrac{AT}{OA} = \dfrac{AT}{1}$ changes from 0 to ∞.

Second quadrant. While θ changes from R to $2R$, $A'T'$ changes from $+\infty$ to $+0$, therefore $\tan\theta = \dfrac{A'T'}{OA'} = \dfrac{A'T'}{-1}$ changes from $-\infty$ to -0.

Third quadrant. While θ changes from $2R$ to $3R$, $A'T'$ changes from -0 to $-\infty$, therefore $\tan\theta = \dfrac{A'T'}{OA'} = \dfrac{A'T'}{-1}$ changes from $+0$ to $+\infty$.

Fourth quadrant. While θ changes from $3R$ to $4R$, AT changes from $-\infty$ to -0, therefore $\tan\theta = \dfrac{AT}{OA} = \dfrac{AT}{1}$ changes from $-\infty$ to -0.

97. Summary of Results. The results of the two preceding articles are brought together in the following table:

Quadrant.	I.	II.	III.	IV.
Angle	0 to R	R to $2R$	$2R$ to $3R$	$3R$ to $4R$
sin	$+0$ to $+1$	$+1$ to $+0$	-0 to -1	-1 to -0
cos	$+1$ to $+0$	-0 to -1	-1 to -0	$+0$ to $+1$
tan	$+0$ to $+\infty$	$-\infty$ to -0	$+0$ to $+\infty$	$-\infty$ to -0
csc	$+\infty$ to $+1$	$+1$ to $+\infty$	$-\infty$ to -1	-1 to $-\infty$
sec	$+1$ to $+\infty$	$-\infty$ to -1	-1 to $-\infty$	$+\infty$ to $+1$
cot	$+\infty$ to $+0$	-0 to $-\infty$	$+\infty$ to $+0$	-0 to $-\infty$

The student should observe, —

(*a*) Every sine and cosine has some value between $+1$ and -1.

(*b*) Every secant and cosecant has some value either greater than $+1$, or less than -1.

(*c*) A tangent or cotangent may have any value whatever.

(*d*) The functions change only at the points between the quadrants, that is, when the angle has one of the values R, $2R$, $3R$, $4R$, etc, and then only when the value of the function is either 0 or ∞.

(*e*) To a given value of a function correspond in general two different angles between 0 and $4R$. To a positive sine correspond two angles, one in the first the other in the second quadrant; to a negative sine correspond two angles, one in the third the other in the fourth quadrant. To a negative tangent correspond two angles, one in the second the other in the fourth quadrant, etc.

98. Fundamental Relations. All the fundamental relations between the functions of an acute angle (Article 12) hold true when the angle is unrestricted in magnitude. The argument is an exact repetition of that used in Article 56. It follows that all trigonometric identities which have been proven for the case when the angle does not exceed a right angle, hold universally, that is, whatever be the magnitude of the angle provided that radical expressions such as $\sqrt{1-\cos^2\theta}$, $\sqrt{1+\tan^2\theta}$, etc., be given the proper sign, $+$ or $-$, depending on the quadrant in which θ lies.

99. Representation of Trigonometric Functions by Lines. Until recent times the trigonometric functions were defined by lines connected with a circle as follows:

Let AOP be any angle, AP the arc which this angle intercepts on a circle drawn with O as a center and any length OA as a radius.

Draw the radius OA' perpendicular to OA, the initial side of the angle. Draw tangents to the circle at A and A', and produce OP to intersect these tangents in T and T' respectively. From P draw the perpendiculars PF and PF' to OA and OA' (produced if necessary) respectively.

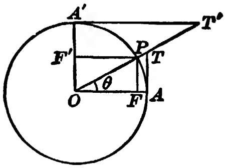


Fig. 133.

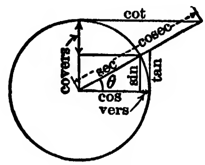


Fig. 134.

The former definitions were then as follows:

$FP =$ sine of arc AP.
$F'P =$ sine of complementary arc $A'P =$ cosine of arc AP.
$AT =$ tangent of arc AP.
$A'T' =$ tangent of complementary arc $A'P =$ cotangent of arc AP.
$OT =$ secant of arc AP.
$OT' =$ secant of complementary arc $A'P =$ cosecant of arc AP.
$FA =$ versine of arc AP.
$F'A' =$ versine of complementary arc $A'P =$ coversine of arc AP.

The definitions just given apply to any arc, provided the conventions regarding the algebraic signs of the various lines be carefully observed. These conventions are, as already stated in Article 53, with the additional one that the distances OT, OT' are positive if they pass through the extremity of the arc in question, that is, if the point P lies between O and T or T'; and negative if they do not, that is, if the point O lies between P and T or T'. Thus for an arc AP in the second quadrant, Fig. 135,

FP, the sine, is positive;
OF, the cosine, is negative;
AT, the tangent, is negative;
$A'T'$, the cotangent, is negative;
OT, the secant, is negative;
OT', the cosecant, is positive.

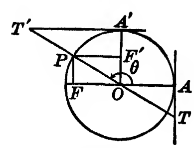


Fig. 135.

According to the old definitions, the length of each of the lines which defines the function depends on the length of the arc; and

since the length of the arc depends on the radius, it was necessary to specify the length of the radius employed. If, however, all lengths are expressed in terms of the radius as unit, the old definitions agree with the modern definitions. Thus in Fig. 133,

$$\tan \operatorname{arc} AP = AT \text{ (old definition).}$$

Now $\dfrac{\operatorname{arc} AP}{OA} = \theta$, the measure of the angle which the arc subtends,

and $\dfrac{AT}{OA} =$ the measure of AT using OA as the unit of measure,

so that, if we substitute for the actual lengths of the arc AP and the line AT their measures in terms of the radius, the old definition becomes

$$\tan \theta = \frac{AT}{OA}, \text{ which is the modern definition.}$$

Similarly, the old and the new definitions of each of the other functions may be shown to agree.

The definitions of the functions as lines, while no longer used as definitions, are still useful in many ways. By their means the variation of the functions in the various quadrants is most readily traced, and the fundamental relations $\sin^2\theta + \cos^2\theta = 1$, $\tan^2\theta + 1 = \sec^2\theta$, $\cot^2\theta + 1 = \csc^2\theta$ become manifest at sight. Above all, they explain the origin of the names of the functions.*

Exercise 46

1. Make out a table giving each of the functions of 45°, 135°, 225°, 315°.

2. Make out a table giving each of the functions of 30°, 150°, 210°, 330°.

* In the light of the historic definitions the origin of the terms *tangent* and *secant* is obvious. The origin of the terms *cosine, cotangent* and *cosecant* has already been explained. The origin of the term *sine* is probably as follows. The Latin word from which the word *sine* is derived is *sinus*, meaning "bay" or "bosom." The Arabic word was *dschiba*, meaning "half the chord of double an arc." Owing to the practice of the Arabs to omit the vowels in writing, *dschiba* was confused with *dschaib* meaning "bay" or "bosom," and it was this word *dschaib* which the Roman translators properly rendered *sinus*. The word *arc* comes from the Latin *arcus*, meaning "a bow." The *versed sine* was formerly called *sagitta*, an arrow, because it occupied the position of the arrow in a bow.

The modern conception of the functions as ratios dates from the second half of the seventeenth century. The old definitions modified by using unity for the radius of the circle were used by many writers less than twenty-five years ago.

3. Given $\sin\theta = \frac{1}{2}$, find the values of θ less than $4R$. What will be the values less than $4R$ which θ may have if $\cos\theta = \frac{1}{2}$?

Ans. $30°, 150°; 60°, 300°$.

4. Find the values of the following functions: $\sin 390°$, $\cos 765°$, $\tan 405°$, $\sin(-45°)$, $\cos(-30°)$, $\tan(-135°)$.

Ans. $\frac{1}{2}, \frac{1}{2}\sqrt{2}, 1, -\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{3}, 1$.

5. Trace the changes of the cosine through each of the four quadrants from the changes of the sine by means of the relation

$$\cos\theta = \sqrt{1 - \sin^2\theta}.$$

6. Trace the changes of the secant in the first quadrant from those of the tangent by means of the relation $\sec\theta = \sqrt{1 + \tan^2\theta}$.

7. Construct the lines representing the various functions of an arc in the third quadrant; of an arc in the fourth quadrant.

8. What sign must be attached to the radical in $\sin\theta = \sqrt{1 - \cos^2\theta}$, when θ is an angle in the second quadrant? In the third quadrant?

9. What sign must be attached to the radical in

$$\sec\theta = \sqrt{1 + \tan^2\theta},$$

when θ is an angle in the third quadrant? In the fourth quadrant?

100. Reduction of Trigonometric Functions to the First Quadrant. In Articles 57 and 58 it was shown how to express any function of an angle in the second quadrant in terms of functions of an angle less than R. It remains to be shown how functions of angles in the third and fourth quadrants may be expressed in terms of functions of angles less than R. When this has been done, the value of any function of any angle can be found from the tables which contain the functions of angles in the first quadrant, that is, from 0° to 90°.

101. Reductions from the Third Quadrant. Any angle θ_3 in the third quadrant lies between $2R$ and $3R$, hence every such angle may be expressed by either

$$\theta_3 = 2R + \theta, \quad \text{or} \quad \theta_3 = 3R - \phi,$$

where θ and ϕ is each less than R,

(*a*) $\theta_3 = 2R + \theta$. Let θ_3 = angle AOP_3 represent any angle in the third quadrant, and let θ = angle $A'OP_3$. Produce P_3O to P, making $OP = OP_3 = r$, then angle $AOP = \theta$. Let (x, y), (x_3, y_3) denote the coördinates of P and P_3 respectively. Draw PF and P_3F_3 perpendicular to AA', then the triangles OPF and OP_3F_3 are geometrically equal and $y_3 = -y$, $x_3 = -x$. Hence we have,—

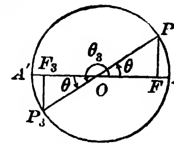


Fig. 136.

$$\left.\begin{aligned}\sin\theta_3 &= \sin(2R+\theta) = \frac{y_3}{r} = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta,\\ \cos\theta_3 &= \cos(2R+\theta) = \frac{x_3}{r} = \frac{-x}{r} = -\frac{x}{r} = -\cos\theta,\\ \tan\theta_3 &= \tan(2R+\theta) = \frac{y_3}{x_3} = \frac{-y}{-x} = \frac{y}{x} = \tan\theta.\end{aligned}\right\} \quad (A)$$

From (A) we obtain

$$\left.\begin{aligned}\csc\theta_3 &= \csc(2R+\theta) = \frac{1}{\sin(2R+\theta)} = \frac{1}{-\sin\theta} = -\csc\theta,\\ \sec\theta_3 &= \sec(2R+\theta) = \frac{1}{\cos(2R+\theta)} = \frac{1}{-\cos\theta} = -\sec\theta,\\ \cot\theta_3 &= \cot(2R+\theta) = \frac{1}{\tan(2R+\theta)} = \frac{1}{\tan\theta} = \cot\theta.\end{aligned}\right\} \quad (A')$$

Observing that in both (A) and (A') the signs on the right are the signs of the functions in the third quadrant, we have the simple rule:

Any function of $(2R + \theta)$ *is equal to plus or minus the same function of* θ, *the sign being that of the function in the third quadrant.*

EXAMPLE. $\sin 204° = \sin(180° + 24°) = -\sin 24° = -0.4067$,
$\cos 204° = \cos(180° + 24°) = -\cos 24° = -0.9135$,
$\tan 204° = \tan(180° + 24°) = \tan 24° = 0.4453$.

(*b*) $\theta_3 = 3R - \phi$.

Put $R - \phi = \theta$, then $\sin\theta = \cos\phi$, $\cos\theta = \sin\phi$, $\tan\theta = \cot\phi$. Hence

$$\left.\begin{aligned}\sin\theta_3 &= \sin(3R-\phi) = \sin(2R+\theta) = -\sin\theta = -\cos\phi,\\ \cos\theta_3 &= \cos(3R-\phi) = \cos(2R+\theta) = -\cos\theta = -\sin\phi,\\ \tan\theta_3 &= \tan(3R-\phi) = \tan(2R+\theta) = \tan\theta = \cot\phi,\end{aligned}\right\} \quad (B)$$

and from (B)

$$\left.\begin{aligned}
\csc\theta_3 &= \csc(3R-\phi) = \frac{1}{\sin(3R-\phi)} = \frac{1}{-\cos\phi} = -\sec\phi,\\
\sec\theta_3 &= \sec(3R-\phi) = \frac{1}{\cos(3R-\phi)} = \frac{1}{-\sin\phi} = -\csc\phi,\\
\cot\theta_3 &= \cot(3R-\phi) = \frac{1}{\tan(3R-\phi)} = \frac{1}{\cot\phi} = \tan\phi.
\end{aligned}\right\} \quad (B')$$

In (B) and (B') the signs on the right are again the signs in the third quadrant of the functions on the left, hence the second rule:

Any function of $(3R-\phi)$ is equal to plus or minus the corresponding cofunction of ϕ, the sign being that of the functions in the third quadrant.

EXAMPLE. $\sin 204^\circ = \sin(270^\circ - 66^\circ) = -\cos 66^\circ = -0.4067,$
$\cos 204^\circ = \cos(270^\circ - 66^\circ) = -\sin 66^\circ = -0.9135,$
$\tan 204^\circ = \tan(270^\circ - 66) = \cot 66 = 0.4453.$

102. Reductions from the Fourth Quadrant. Any angle θ_4 in the fourth quadrant lies between $3R$ and $4R$, hence every such angle can be expressed by either

$$\theta_4 = 4R - \theta, \qquad \text{or} \qquad \theta_4 = 3R + \phi,$$

where θ and ϕ are angles less than R.

(a) $\theta_4 = 4R - \theta$. Let $\theta_4 =$ angle AOP_4 be any angle in the fourth quadrant, and let us put angle $P_4OA = \theta$. Draw $OP = OP_4 = r$, making angle $AOP = \theta$. From P and P_4 draw perpendiculars to OA. Then the triangles OPF and OP_4F are geometrically equal, and if (x, y), (x_4, y_4) denote the coördinates of P and P_4 respectively, we have $y_4 = -y$, $x_4 = x$.

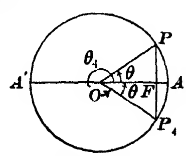


Fig. 137.

Consequently

$$\left.\begin{aligned}
\sin\theta_4 &= \sin(4R-\theta) = \frac{y_4}{r} = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta,\\
\cos\theta_4 &= \cos(4R-\theta) = \frac{x_4}{r} = \frac{x}{r} = \cos\theta,\\
\tan\theta_4 &= \tan(4R-\theta) = \frac{y_4}{x_4} = \frac{-y}{x} = -\frac{y}{x} = -\tan\theta.
\end{aligned}\right\} \quad (A)$$

From (A) follows

$$\left.\begin{aligned}\csc\theta_4 &= \csc(4R-\theta) = \frac{1}{\sin(4R-\theta)} = \frac{1}{-\sin\theta} = -\csc\theta,\\ \sec\theta_4 &= \sec(4R-\theta) = \frac{1}{\cos(4R-\theta)} = \frac{1}{\cos\theta} = \sec\theta,\\ \cot\theta_4 &= \cot(4R-\theta) = \frac{1}{\tan(4R-\theta)} = \frac{1}{-\tan\theta} = -\cot\theta.\end{aligned}\right\}\ (A')$$

In (A) and (A') the signs on the right are the signs of the functions in the fourth quadrant, hence

Any function of $(4R-\theta)$ is equal to plus or minus the same function of θ, the sign being that of the function in the fourth quadrant.

EXAMPLE.
$$\begin{aligned}\sin\frac{11\pi}{6} &= \sin\left(2\pi-\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\tfrac{1}{2},\\ \cos\frac{11\pi}{6} &= \cos\left(2\pi-\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \tfrac{1}{2}\sqrt{3},\\ \tan\frac{11\pi}{6} &= \tan\left(2\pi-\frac{\pi}{6}\right) = -\tan\frac{\pi}{6} = -\tfrac{1}{3}\sqrt{3}.\end{aligned}$$

(b) $\theta_4 = 3R+\phi$.

Put $\phi = R-\theta$, then $\sin\phi = \cos\theta$, $\cos\phi = \sin\theta$, $\cot\phi = \tan\theta$, hence

$$\left.\begin{aligned}\sin\theta_4 &= \sin(3R+\phi) = \sin(4R-\theta) = -\sin\theta = -\cos\phi,\\ \cos\theta_4 &= \cos(3R+\phi) = \cos(4R-\theta) = \cos\theta = \sin\phi,\\ \tan\theta_4 &= \tan(3R+\phi) = \tan(4R-\theta) = -\tan\theta = -\cot\phi,\end{aligned}\right\}\ (B)$$

and from (B)

$$\left.\begin{aligned}\csc\theta_4 &= \csc(3R+\phi) = \frac{1}{\sin(3R+\phi)} = \frac{1}{-\cos\phi} = -\sec\phi,\\ \sec\theta_4 &= \sec(3R+\phi) = \frac{1}{\cos(3R+\phi)} = \frac{1}{\sin\phi} = \csc\phi,\\ \cot\theta_4 &= \cot(3R+\phi) = \frac{1}{\tan(3R+\phi)} = \frac{1}{-\cot\phi} = -\tan\phi.\end{aligned}\right\}\ (B')$$

In (B) and (B') the signs on the right are the signs in the fourth quadrant of the functions on the left, hence

Any function of $(3R+\phi)$ is equal to plus or minus the corresponding cofunction of ϕ, the sign being that of the function in the fourth quadrant.

EXAMPLE. $\sin\frac{11\pi}{6} = \sin\left(\frac{3\pi}{2}+\frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2},$

$$\cos\frac{11\pi}{6} = \cos\left(\frac{3\pi}{2}+\frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \tfrac{1}{2}\sqrt{3},$$

$$\tan\frac{11\pi}{6} = \tan\left(\frac{3\pi}{2}+\frac{\pi}{3}\right) = -\cot\frac{\pi}{3} = -\tfrac{1}{3}\sqrt{3}.$$

103. Functions of Negative Angles.

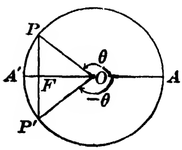


Fig. 138.

Let $-\theta$ = angle AOP' be any negative angle. Construct the angle $AOP = \theta$. Take $OP = OP' = r$, and let (x, y), (x', y') denote the coördinates of the points P and P' respectively, then $x' = x$, $y' = -y$, and we have

$$\left.\begin{aligned} \sin(-\theta) &= \frac{y'}{r} = \frac{-y}{r} = -\sin\theta, \\ \cos(-\theta) &= \frac{x'}{r} = \frac{x}{r} = \cos\theta, \\ \tan(-\theta) &= \frac{y'}{x'} = \frac{-y}{x} = -\tan\theta, \end{aligned}\right\} \qquad (A)$$

and from (A)

$$\left.\begin{aligned} \csc(-\theta) &= \frac{1}{\sin(-\theta)} = \frac{1}{-\sin\theta} = -\csc\theta, \\ \sec(-\theta) &= \frac{1}{\cos(-\theta)} = \frac{1}{\cos\theta} = \sec\theta, \\ \cot(-\theta) &= \frac{1}{\tan(-\theta)} = \frac{1}{-\tan\theta} = -\cot\theta. \end{aligned}\right\} \qquad (A')$$

The signs on the right are the signs of the functions in the fourth quadrant, hence

Any function of $(-\theta)$ is equal to plus or minus the same function of θ, the sign being that of the function in the fourth quadrant.

EXAMPLE. $\sin(-13°\ 25') = -\sin 13°\ 25' = -0.2320,$

$$\cos(-13°\ 25') = \cos 13°\ 25' = 0.9727,$$

$$\tan(-13°\ 25') = -\tan 13°\ 25' = -0.2385.$$

104. Table of Principal Reduction Formulas and General Rules.

The principal results of the last three articles, together with the corresponding results for the first and second quadrants (Articles 10, 57, 58), are brought together in the following table for purposes of comparison and reference.

Quadrant I

$$\sin(R-\phi)=\cos\phi,\quad \cos(R-\phi)=\sin\phi,\quad \tan(R-\phi)=\cot\phi.$$

Quadrant II

$\sin(2R-\theta)=\sin\theta,$	$\sin(R+\phi)=\cos\phi,$
$\cos(2R-\theta)=-\cos\theta,$	$\cos(R+\phi)=-\sin\phi,$
$\tan(2R-\theta)=-\tan\theta.$	$\tan(R+\phi)=-\cot\phi.$

Quadrant III

$\sin(2R+\theta)=-\sin\theta,$	$\sin(3R-\phi)=-\cos\phi,$
$\cos(2R+\theta)=-\cos\theta,$	$\cos(3R-\phi)=-\sin\phi,$
$\tan(2R+\theta)=\tan\theta.$	$\tan(3R-\phi)=\cot\phi.$

Quadrant IV

$\sin(4R-\theta)=\sin(-\theta)=-\sin\theta,$	$\sin(3R+\phi)=-\cos\phi,$
$\cos(4R-\theta)=\cos(-\theta)=\cos\theta,$	$\cos(3R+\phi)=\sin\phi,$
$\tan(4R-\theta)=\tan(-\theta)=-\tan\theta.$	$\tan(3R+\phi)=-\cot\phi.$

We observe that each equation on the left involves a pair of same-named functions and the coefficients of R are even numbers, 2 or 4. On the right each equation involves a pair of conamed functions and the coefficients of R are the odd numbers 1 and 3. In either case the signs are the signs of the functions in the respective quadrants.

By increasing the angles on the left by multiples of $4R$ (which will not change the value of the functions), we obtain formulas for the functions of

$$6R-\theta,\ 6R+\theta,\ 8R-\theta,\ 8R+\theta,\ \ldots,\ \ldots,\ \ldots,\ 2nR\pm\theta,$$

and by increasing the angles on the right by multiples of $4R$, we obtain

$$5R+\phi,\ 5R-\phi,\ 7R+\phi,\ 7R-\phi,\ \ldots,\ \ldots,\ \ldots,\ (2n+1)R\pm\phi.$$

All the foregoing results are therefore included in the two formulas,

$$\textit{Any function } (2\,nR \pm \theta) = \pm \textit{ same function } \theta,$$

$$\textit{Any function } (\overline{2\,n + 1\,R} \pm \phi) = \pm \textit{ cofunction } \phi,$$

the sign being the sign of the function on the left in the quadrant in which the angle falls when θ or ϕ are acute angles.*

Exercise 47

1. Express in terms of same-named functions of angles less than R,

$\sin 146°$, $\cos 235°$, $\tan 317°$, $\sin\frac{5\pi}{4}$, $\cos\frac{7\pi}{8}$, $\tan\frac{15\pi}{4}$.

Ans. $\sin 34°$, $-\cos 55°$, $-\tan 43°$, $-\sin\frac{\pi}{4}$, $-\cos\frac{\pi}{8}$, $-\tan\frac{\pi}{4}$.

2. Express in terms of cofunctions of angles less than R,

$\tan 95°$, $\sin 272°$, $\cos 115°\,10'$, $\sec\frac{2\pi}{3}$, $\sin\frac{28\pi}{3}$, $\cot 3\frac{1}{2}R$.

Ans. $-\cot 5°$, $-\cos 2°$, $-\sin 25°\,10'$, $-\csc\frac{\pi}{6}$, $-\cos\frac{\pi}{6}$, $-\tan\frac{R}{2}$.

3. Express in terms of functions of positive angles less than $45°$ $\sin 143°\,15'$, $\cos 143°\,15'$, $\tan 243°\,10'\,15''$, $\sec 284°\,30'$, $\cot 127°$.
Ans. $\sin 36°\,45'$, $-\cos 36°\,45'$, $\cot 26°\,49'\,45''$, $\csc 14°\,30'$, $-\tan 37°$

4. Use natural functions table to find

$\sin 111°\,30'$, $\cos 253°\,12'$, $\tan 134°$, $\sin 317°\,15'$, $\cos 97°\,35'$.

Ans. 0.9304, $-$ 0.2890, $-$ 1.0355, $-$ 0.6788, $-$ 0.1320.

5. Find $\sin(-150°)$, $\cos 3564°$, $\tan(-5445°)$, $\sin\left(-27\frac{\pi}{4}\right)$, $\cos(-100°)$. *Ans.* $-\frac{1}{2}$, 0.8090, -1, $-\frac{1}{2}\sqrt{2}$, $-$ 0.1736.

6. If $\sin\theta = 0.5831$, what values less than $4\,R$ may θ have?

7. If $\tan\theta = -4.3897$, and $\sin\theta$ is known to be positive, find the value of θ. *Ans.* $\theta = 102°\,50'$.

8. If $\cos\theta = \sin 147°$, show that one value of θ is $303°$.

9. If $\sin\theta = \cos 5\,\theta$, show that one value of θ is $15°$, and another $75°$.

10. Given $\sin\phi = -0.4561$; find $\tan\phi$. *Ans.* $\tan\phi = 0.5125$.

* These rules hold not only for the sine, cosine and tangent, but for the cosecant, secant and cotangent as well. The latter have been omitted from the summary on account of their lesser importance.

105. Generalization of the Preceding Reduction Formulas. In the proof for the formulas for the functions of a negative angle (Article 103), θ was not restricted in magnitude. These formulas therefore hold true for any angle, but in the formulas for the functions of $2R + \theta$, $3R - \phi$ (Article 101), and of $4R - \theta$, $3R + \phi$ (Article 102), θ and ϕ were assumed to be angles between 0 and R. This restriction is not necessary and will now be removed. In other words, we will now show that the formulas of Articles 101 and 102 hold for any value of θ and ϕ. The complete list of reduction formulas includes the formulas for the functions of $2R - \theta$ and of $R \pm \phi$, we shall show that these formulas also hold for any value of the angles.

For the sake of brevity the proofs will be confined to the first one of the formulas in each set. The proofs of the other formulas are left as exercises for the student.

(*a*) *Functions of* $(2R + \theta)$. The angles $(2R + \theta)$ and θ differ by $2R$ no matter how large θ is and whether θ is positive or negative. Consequently the points P_3 and P (Fig. 136) must always lie on a straight line through the origin. The coördinates of P_3 and P will therefore be numerically equal but opposite in sign. Hence for any value of θ, positive or negative,

$$\sin(2R + \theta) = \frac{y_3}{r} = \frac{-y}{r} = -\sin\theta. \tag{1}$$

This establishes the first of the relations (A), Article 101, for every value of θ.

(*b*) *Functions of* $(2R - \theta)$. If in (1) we put for θ, $-\theta$, we obtain

$$\sin(2R - \theta) = -\sin(-\theta).$$

But by (A), Article 103, $\sin(-\theta) = -\sin\theta$ for every value of θ, therefore

$$\sin(2R - \theta) = \sin\theta. \tag{2}$$

This establishes the first of the relations in Article 57 for every value of θ.

(*c*) *Functions of* $(4R - \theta)$. The functions of an angle are not changed if the angle is increased or diminished by $4R$ (Article 94), hence

$$\sin(4R - \theta) = \sin(-\theta), \text{ for every value of } \theta.$$

But $\sin(-\theta) = -\sin\theta$, by Article 103,

therefore $\sin(4R - \theta) = -\sin\theta$. (3)

This establishes the first of the relations (*A*), Article 102, for every value of the angle.

(*d*) *Functions of* $(R - \phi)$. Let ϕ be any angle and let ϕ' be the smallest positive angle coterminal with ϕ. Then ϕ' can be written in one of the forms

θ, $2R - \theta$, $2R + \theta$, $4R - \theta$, where θ is positive and less than R, according as ϕ' is an angle in the first, second, third or fourth quadrant.

If $\phi' = 2R - \theta$, then

$$\sin(R - \phi') = \sin(-R + \theta) = -\sin(R - \theta) = -\cos\theta = \cos\phi'.$$

If $\phi' = 2R + \theta$, then

$$\sin(R - \phi') = \sin(-R - \theta) = -\sin(R + \theta) = -\cos\theta = \cos\phi'.$$

If $\phi' = 4R - \theta$, then

$$\sin(R - \phi') = \sin(-3R + \theta) = -\sin(3R - \theta) = \cos\theta = \cos\phi'.$$

We see then that, whether ϕ' is in the first, second, third or fourth quadrant,

$$\sin(R - \phi') = \cos\phi',$$

and hence

$$\sin(R - \phi) = \cos\phi \qquad (4)$$

is established for every value of ϕ.

(*e*) *Functions of* $(R + \phi)$, $(3R - \phi)$, $(3R + \phi)$. Let ϕ be any angle and put $\phi = R - \theta$. Then by applying the formulas whose generality has been already established, we find

$$\sin(R + \phi) = \sin(2R - \theta) = \sin\theta = \cos\phi \qquad (5)$$

$$\sin(3R - \phi) = \sin(2R + \theta) = -\sin\theta = -\cos\phi, \qquad (6)$$

$$\sin(3R + \phi) = \sin(4R - \theta) = -\sin\theta = -\cos\phi. \qquad (7)$$

Exercise 48

Repeat the argument of the preceding article to show that for all values of θ and ϕ respectively,—

1. $\tan(2R \pm \theta) = \pm\tan\theta$.
2. $\tan(R \pm \phi) = \mp\cot\phi$.
3. $\tan(3R \pm \phi) = \pm\cot\phi$.

4. Show that for every value of θ,

$$\sin(\theta - R) = -\cos\theta,\ \sin(\theta - 2R) = -\sin\theta,\ \sin(\theta - 3R) = \cos\theta.$$
$$\cos(\theta - R) = \sin\theta,\ \cos(\theta - 2R) = -\cos\theta,\ \cos(\theta - 3R) = -\sin\theta.$$
$$\tan(\theta - R) = -\cot\theta,\ \tan(\theta - 2R) = \tan\theta,\ \tan(\theta - 3R) = -\cot\theta.$$

5. Prove geometrically that $\cos(R - \phi) = \sin\phi$,

(*a*) When ϕ is an angle in the second quadrant.
(*b*) When ϕ is an angle in the third quadrant.
(*c*) When ϕ is an angle in the fourth quadrant.

6. General proofs of the reduction formulas for the angles

$$(2R - \theta),\quad (4R - \theta),\quad (2R + \theta)$$

may be obtained from considerations of symmetry. Referred to the same origin and taking the initial line to coincide with the x-axis, then no matter how large θ, and whether positive or negative,

(*a*) The terminal sides of θ and $(2R - \theta)$ are symmetrically situated with respect to the y-axis.
(*b*) The terminal sides of θ and $(4R - \theta)$ are symmetrically situated with respect to the x-axis.
(*c*) The terminal sides of θ and $(2R + \theta)$ are symmetrically situated with respect to the origin.

It follows that the same functions of each pair of angles are numerically equal and that in

(*a*) The sines have like and the cosines opposite signs.
(*b*) The sines have opposite and the cosines like signs.
(*c*) The sines and cosines each have opposite signs.

In each case the sign of the tangent may be determined from the relation

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

CHAPTER XI

FUNCTIONS OF TWO OR MORE ANGLES

106. Addition Theorems for the Sine and Cosine.

First Proof. If a, b, c denote the sides of any triangle and A, B, C the angles opposite these sides, we have from the law of sines (Article 62, (3)),

$$a = D \sin A,$$
$$b = D \sin B,$$
$$c = D \sin C,$$

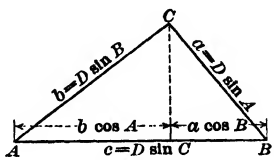


Fig. 139.

and by the projection theorem (Article 63, (4)),

$$c = a \cos B + b \cos A.$$

Substituting in this equation the above values for a, b and c, and dividing out the constant factor D, we get

$$\sin C = \sin A \cos B + \cos A \sin B.$$

Now $C = 180° - (A + B)$, therefore $\sin C = \sin (A + B)$, whence

$$\mathbf{\sin (A + B) = \sin A \cos B + \cos A \sin B.} \qquad (1)$$

Again

$$\begin{aligned} \cos^2 (A + B) &= 1 - \sin^2 (A + B) \\ &= 1 - (\sin A \cos B + \cos A \sin B)^2 \\ &= 1 - \sin^2 A \cos^2 B - 2 \sin A \sin B \cos A \cos B - \cos^2 A \sin^2 B \\ &= 1 - (1 - \cos^2 A) \cos^2 B - 2 (\; . \; . \; . \;) - (1 - \sin^2 A) \sin^2 B \\ &= \cos^2 A \cos^2 B - 2 \sin A \sin B \cos A \cos B + \sin^2 A \sin^2 B \\ &= (\cos A \cos B - \sin A \sin B)^2. \end{aligned}$$

Taking the square root of both sides,

$$\mathbf{\cos (A + B) = \cos A \cos B - \sin A \sin B.} \qquad (2)$$

Since the last expression was obtained by extracting a square root it may seem that the double sign, $\pm$, should have been put before the right-hand member of (2), but on putting $B = 0$, the minus

would yield the result $\cos A = -\cos A$, which shows that the minus sign cannot be used.

Formulas (1) and (2) embody the so-called *addition theorems* for the sine and cosine respectively. In words,—

The sine of the sum of two angles is equal to the sine of the first angle times the cosine of the second plus the cosine of the first angle times the sine of the second.

The cosine of the sum of two angles is equal to the product of the cosines of the separate angles diminished by the product of their sines.

It is plain that by means of these theorems the sine and cosine of the sum of two angles may be found if the sines and cosines of each of the separate angles are known.

EXAMPLE 1. Given the functions of 45° and of 30°, to find the sine and cosine of 75°.

Solution.
$$\sin 75^\circ = \sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$
$$= \tfrac{1}{2}\sqrt{2}\cdot\tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\sqrt{2}\cdot\tfrac{1}{2} = \tfrac{1}{4}(\sqrt{6}+\sqrt{2}).$$
$$\cos 75^\circ = \cos(45^\circ + 30^\circ) = \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$$
$$= \tfrac{1}{2}\sqrt{2}\cdot\tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}\sqrt{2}\cdot\tfrac{1}{2} = \tfrac{1}{4}(\sqrt{6}-\sqrt{2}).$$

107. Generalization of the Addition Theorems. In the foregoing demonstration, A and B are angles of a triangle, their sum is therefore necessarily less than $2R$. This restriction may be removed, in other words, the addition theorems hold for angles of any magnitude and whether positive or negative.

To prove this let A and B be two angles each less than R, so that their sum is less than $2R$, and let $A_1 = A \pm R$, then

$$\sin A_1 = \pm\cos A,\ \cos A_1 = \mp\sin A, \text{ (Art. 104, and Ex. 48, 4)}$$
$$\begin{aligned}
\sin(A_1+B) &= \sin(A \pm R + B) = \pm\cos(A+B)\\
&= \pm(\cos A\cos B - \sin A \sin B)\\
&= \pm\cos A\cos B \mp \sin A\sin B\\
&= \sin A_1\cos B + \cos A_1 \sin B. \qquad (1)\\
\cos(A_1+B) &= \cos(A \pm R + B) = \mp\sin(A+B)\\
&= \mp(\sin A\cos B + \cos A\sin B)\\
&= \mp\sin A\cos B \mp \cos A\sin B\\
&= \cos A_1\cos B - \sin A_1\sin B\,^* \qquad (2)
\end{aligned}$$

* If the student finds it difficult to follow the double signs, let him consider the two cases $A_1 = A + R$, $A_1 = A - R$, separately.

Equations (1) and (2) show that the addition theorems continue to hold if the angle A is increased or diminished by R, and the same reasoning applies to the angle B. By a repetition of the process just employed it is clear that the theorems will continue to hold true if A_1 is replaced by A_2, where $A_2 = A_1 \pm R = A_1 \pm 2R$, and generally that A may be replaced by

$$A_n = A \pm nR,$$

and B by

$$B_m = B \pm mR,$$

n and m being two arbitrary integers.

But this proves the theorems for all values of the angles, for any positive or negative angle may be put in the form $A \pm nR$, where n is some integer and A some angle less than R.

108. Addition Theorems. *Second Proof.* The addition theorems may be proved without making use of the law of sines and the projection theorem.

In Fig. 140, let XOM = angle A, and MON = angle B, then XON = angle $(A + B)$.

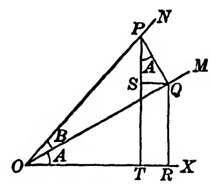


Fig. 140.

On ON take any point P, and from P draw PT and PQ perpendicular to OX and OM respectively.

From Q draw QR perpendicular to OX and QS parallel to OX.

The triangles QOR and QPS are similar (Why?), hence angle QPS = angle A. Now

$$\begin{aligned}\sin(A+B) &= \frac{TP}{OP} = \frac{TS + SP}{OP} = \frac{RQ}{OP} + \frac{SP}{OP}\\ &= \frac{RQ}{OQ}\cdot\frac{OQ}{OP} + \frac{SP}{QP}\cdot\frac{QP}{OP}\\ &= \sin A\cos B + \cos A\sin B.\end{aligned}$$

$$\begin{aligned}\cos(A+B) &= \frac{OT}{OP} = \frac{OR - TR}{OP} = \frac{OR}{OP} - \frac{SQ}{OP}\\ &= \frac{OR}{OQ}\cdot\frac{OQ}{OP} - \frac{SQ}{QP}\cdot\frac{QP}{OP}\\ &= \cos A\cos B - \sin A\sin B.\end{aligned}$$

In the figure we have taken $A + B$ less than a right angle, but the proof just given will hold for any angles, provided proper attention be given to the algebraic signs of the lines which enter the figure.

109. Subtraction Theorems for the Sine and Cosine. Since the addition theorems have been shown to hold for negative as well as for positive angles, we may replace B by $-B$. The equations (1) and (2), Article 106, then become

$$\sin(A - B) = \sin A \cos(-B) + \cos A \sin(-B)$$
$$\cos(A - B) = \cos A \cos(-B) - \sin A \sin(-B),$$

from which

$$\mathbf{\sin(A - B) = \sin A \cos B - \cos A \sin B,} \qquad (1)$$
$$\mathbf{\cos(A - B) = \cos A \cos B + \sin A \sin B.} \qquad (2)$$

These formulas enable us to compute the sine and cosine of the difference of two angles if the sines and cosines of the separate angles are known.

EXAMPLE. Given the functions of $45°$ and $30°$; to find the sine and cosine of $15°$.

Solution.

$$\sin 15° = \sin(45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30°$$
$$= \tfrac{1}{2}\sqrt{2} \cdot \tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}\sqrt{2} \cdot \tfrac{1}{2} = \tfrac{1}{4}(\sqrt{6} - \sqrt{2}).$$
$$\cos 15° = \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$$
$$= \tfrac{1}{2}\sqrt{2} \cdot \tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\sqrt{2} \cdot \tfrac{1}{2} = \tfrac{1}{4}(\sqrt{6} + \sqrt{2}).$$

EXERCISE 49

1. Find the sine and cosine of $15°$ from the relation, $15° = 60° - 45°$.

2. Given $\sin x = \frac{3}{5}$, $\cos x = \frac{4}{5}$, $\sin y = \frac{8}{17}$, $\cos y = \frac{15}{17}$; find $\sin(x + y)$ and $\cos(x + y)$.

3. Find $\sin 90°$ and $\cos 90°$ from the relation $90° = 60° + 30°$.

4. Find $\sin 0°$ and $\cos 0°$ from the relation $0° = 30° - 30°$.

5. Apply the addition and subtraction theorems to find the following:

$$\sin(90° - x),\quad \cos(90° + x),\ \sin(180° + x),\ \cos(270° - x),$$
$$\sin(360° - y),\ \cos(45° - y),\ \sin(30° + y),\quad \cos(60° - y).$$

6. Show that

$$\sin(A+B)+\sin(A-B) = 2\sin A\cos B,$$
$$\sin(A+B)-\sin(A-B) = 2\cos A\sin B,$$
$$\cos(A+B)+\cos(A-B) = 2\cos A\cos B,$$
$$\cos(A+B)-\cos(A-B) = -2\sin A\sin B.$$

7. By putting $B = A$ in Problem 6 show that

$$\sin 2A = 2\sin A\cos A, \cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A.$$

8. Prove the subtraction theorems geometrically by means of Fig. 141. $XOQ =$ angle A, $MON =$ angle B, $XON =$ angle $(A-B)$. QR, QP, PT are perpendiculars to OX, OM, OX respectively. Angle $SQP =$ angle A (Why?).

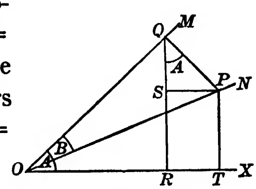


Fig. 141.

9. Show that the answers to Problem 17, Exercise 38, may be put in the forms

$$x = \frac{a\sin\beta\cos\alpha}{\sin(\alpha-\beta)}, \qquad y = \frac{a\sin\alpha\sin\beta}{\sin(\alpha-\beta)}.$$

10. Show that the answer to Problem 19, Exercise 38, may be written

$$h = \frac{a\sin\alpha\sin\alpha'}{\sqrt{\sin(\alpha-\alpha')\sin(\alpha+\alpha')}}.$$

11. Show that

$$\tan A \pm \tan B = \frac{\sin(A\pm B)}{\cos A\cos B},$$

and hence that

$$\frac{\tan A+\tan B}{\tan A-\tan B} = \frac{\sin(A+B)}{\sin(A-B)}.$$

12. Show that $\sin(n+1)\theta = \sin n\theta\cos\theta + \cos n\theta\sin\theta$,

and hence that $\sin\overline{n+1}^\circ = \sin n^\circ\cos 1^\circ + \cos n^\circ\sin 1^\circ$,

similarly $\cos\overline{n+1}^\circ = \cos n^\circ\cos 1^\circ - \sin n^\circ\sin 1^\circ$.

Hence, if the sine and cosine of 1° are known, those of 2°, 3°, etc., may be readily computed.

13. Show that $\cos(A+\frac{1}{4}\pi)+\sin(A-\frac{1}{4}\pi) = 0$,

and $\sin(A+\frac{1}{4}\pi)+\cos(A-\frac{1}{4}\pi) = \sqrt{2}(\sin A+\cos A)$.

14. Show that

$$\cos A \sin (B - C) + \cos B \sin (C - A) + \cos C \sin (A - B) = 0,$$
$$\sin A \sin (B - C) + \sin B \sin (C - A) + \sin C \sin (A - B) = 0.$$

15. Two straight roads OA and OB (Fig. 142) cross at an angle α. From O, their point of intersection, a straight road is to be laid out to a point P, which is p miles from the first road and q miles from the second. Required the angle θ which OP will make with OA.

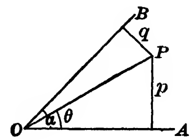


Fig. 142.

$$\frac{p}{q} = \frac{\sin \theta}{\sin (\alpha - \theta)}, \qquad \text{whence} \quad \tan \theta = \frac{p \sin \alpha}{p \cos \alpha + q}.$$

16. The area T of a triangle was computed from two sides, b and c, and the included angle A. Afterwards it was found that an error ϵ had been made in measuring the angle A. Show that the corrected area is given by the formula

$$T' = T (\cos \epsilon + \sin \epsilon \cot A).$$

17. Two parallel forces p and q act on levers of lengths a and b respectively, which are inclined at an angle α at the common fulcrum O. What angle θ must the forces make with the lever a in order that there may be equilibrium?

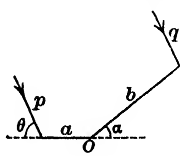


Fig. 143.

(Suggestion. Equating moments about O we have

$$ap \sin \theta = bq \sin (\alpha + \theta),$$

from which
$$\tan \theta = \frac{bq \sin \alpha}{ap - bq \cos \alpha}.)$$

18. Show that

$$\sin (A + B) \sin (A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A,$$
$$\cos (A + B) \cos (A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A.$$

19. x, y, z being any angles, show that

$$\sin (x + y + z) = \sin x \cos y \cos z + \sin y \cos z \cos x + \sin z \cos x \cos y - \sin x \sin y \sin z.$$

$$\cos (x + y + z) = \cos x \cos y \cos z - \cos x \sin y \sin z - \cos y \sin z \sin x - \cos z \sin x \sin y.$$

20. Show that

$$\sin(x+y-z)+\sin(x-y+z)+\sin(-x+y+z) = \sin(x+y+z)+4\sin x\sin y\sin z.$$

$$\cos(x+y-z)+\cos(x-y+z)+\cos(-x+y+z) = 4\cos x\cos y\cos z-\cos(x+y+z).$$

21. By eliminating a, b, c from the equations (Art. 63),

$$c = a\cos B + b\cos A,$$
$$a = b\cos C + c\cos B,$$
$$b = c\cos A + a\cos C,$$

show that

$$\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C = 1.$$

Solve this equation for $\cos C$, and obtain

$$\cos C = -\cos A\cos B \pm \sin A\sin B.$$

Remembering that $C = 180° - (A+B)$, and disregarding the lower sign which is inadmissible (Why?), we find

$$\cos(A+B) = \cos A\cos B - \sin A\sin B.$$

This constitutes another *proof of the addition theorem* for the cosine.

110. Tangent of the Sum and Difference of Two Angles. If we divide the sine of the sum of two angles by the cosine we obtain the tangent, thus

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}.$$

On dividing both the numerator and denominator of the right-hand member by $\cos A\cos B$, we have

$$\tan(A+B) = \frac{\dfrac{\sin A\cos B}{\cos A\cos B}+\dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B}-\dfrac{\sin A\sin B}{\cos A\cos B}} = \frac{\dfrac{\sin A}{\cos A}+\dfrac{\sin B}{\cos B}}{1-\dfrac{\sin A}{\cos A}\dfrac{\sin B}{\cos B}}$$

that is

$$\mathbf{\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}}. \qquad (1)$$

To obtain the tangent of the difference of two angles, we need only put in (1) for B, $-B$, thus

$$\tan(A-B) = \frac{\tan A+\tan(-B)}{1-\tan A\tan(-B)},$$

that is,

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}. \tag{2}$$

Of course we might have deduced (2) just as we deduced (1) that is, by dividing $\sin(A - B)$ by $\cos(A - B)$.

111. Functions of Double an Angle. If $B = A$, the formulas for the sine, cosine and tangent of the sum of two angles become

$$\sin(A + A) = \sin A \cos A + \cos A \sin A,$$
$$\cos(A + A) = \cos A \cos A - \sin A \sin A,$$
$$\tan(A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A},$$

that is,

$$\sin 2A = 2 \sin A \cos A, \tag{1}$$
$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1, \tag{2}$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}. \tag{3}$$

By means of these formulas the functions of twice an angle are easily computed, provided the functions of the single angle are known.

112. Functions of Half an Angle. It is frequently necessary to express the functions of half an angle in terms of the functions of the whole angle. This is most easily accomplished by means of (2), Article 111. Since these formulas hold for any value of the angle, we may replace A by $\frac{1}{2}A$, thus

$$\cos(2 \cdot \tfrac{1}{2}A) = 1 - 2\sin^2 \tfrac{1}{2}A = 2\cos^2 \tfrac{1}{2}A - 1,$$

or

$$\left.\begin{aligned} \cos A &= 1 - 2\sin^2 \tfrac{1}{2}A \\ &= 2\cos^2 \tfrac{1}{2}A - 1. \end{aligned}\right\} \tag{1}$$

If we solve the first of these equations for $\sin \frac{1}{2}A$, we obtain

$$\sin \tfrac{1}{2}A = \pm\sqrt{\frac{1 - \cos A}{2}}, \tag{2}$$

and the second solved for $\cos \frac{1}{2}A$ gives

$$\cos \tfrac{1}{2}A = \pm\sqrt{\frac{1 + \cos A}{2}}. \tag{3}$$

Dividing (2) by (3) gives

$$\tan \tfrac{1}{2} A = \pm\sqrt{\frac{1 - \cos A}{1 + \cos A}}. \qquad (4)$$

Exercise 50

1. Given the tangents of 45° and 30°, compute the tangents of 75° and 15°.

$$Ans.\quad \tan 75° = \frac{3 + \sqrt{3}}{3 - \sqrt{3}}, \quad \tan 15° = \frac{3 - \sqrt{3}}{3 + \sqrt{3}}.$$

2. Given $\tan A = \frac{1}{2}$, $\tan B = \frac{1}{3}$, find $\tan(A + B)$ and $\tan(A - B)$.

3. Show that

$$\tan(45° + A) = \frac{1 + \tan A}{1 - \tan A}, \qquad \tan(45° - A) = \frac{1 - \tan A}{1 + \tan A}.$$

4. Show that

$$\cot(A + B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}, \qquad \cot(A - B) = \frac{\cot A \cot B + 1}{-\cot A + \cot B}.$$

5. Show that

$$\tan(x + y + z) = \frac{\tan x + \tan y + \tan z - \tan x \tan y \tan z}{1 - \tan x \tan y - \tan y \tan z - \tan z \tan x}.$$

6. Express $\sin 4A$, $\cos 4A$, $\tan 4A$ in terms of the functions of $2A$.

7. Given the functions of 30°; find the sine, cosine and tangent of 60°.

8. Given the functions of 45°; find the sine, cosine and tangent of $22\frac{1}{2}°$.

Ans. $\sin 22\frac{1}{2}° = \frac{1}{2}\sqrt{2 - \sqrt{2}}$, $\cos 22\frac{1}{2}° = \frac{1}{2}\sqrt{2 + \sqrt{2}}$, $\tan 22\frac{1}{2}° = \sqrt{2} - 1$.

9. Given the functions of 30°; find the sine, cosine and tangent of 15°.

Ans. $\sin 15° = \frac{1}{2}\sqrt{2 - \sqrt{3}}$, $\cos 15° = \frac{1}{2}\sqrt{2 + \sqrt{3}}$, $\tan 15° = 2 - \sqrt{3}$.

10. Express $\sin 3A$ in terms of $\sin A$.

Ans. $\sin 3A = 3 \sin A - 4 \sin^3 A$.

(Suggestion. $3A = 2A + A$.)

11. Express $\cos 3A$ in terms of $\cos A$.

Ans. $\cos 3A = 4 \cos^3 A - 3 \cos A$.

12. Express $\tan 3A$ in terms of $\tan A$.

Ans. $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$.

13. Find the sine and cosine of 18°.

Ans. $\sin 18° = \dfrac{\sqrt{5} - 1}{4}$, $\cos 18° = \dfrac{\sqrt{10 + 2\sqrt{5}}}{4}$.

(Suggestion. Let $x = 18°$, then $2x + 3x = 90°$, $2x = 90° - 3x$, $\sin 2x = \cos 3x$.

Now express $\sin 2x$ and $\cos 3x$ each in terms of functions of x, and solve for $\sin x$. Then $\cos x = \sqrt{1 - \sin^2 x}$.)

14. Show that if t stands for $\tan A$,

$$\sin 2A = \frac{2t}{1 + t^2}, \qquad \cos 2A = \frac{1 - t^2}{1 + t^2}, \qquad \tan 2A = \frac{2t}{1 - t^2}.$$

15. Use Fig. 144 to prove that

$$\sin 2A = 2\sin A\cos A,$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

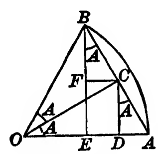


Fig. 144.

16. Use Fig. 145 to prove that

$$\sin \tfrac{1}{2}A = \sqrt{\tfrac{1}{2}(1 - \cos A)},$$

$$\cos \tfrac{1}{2}A = \sqrt{\tfrac{1}{2}(1 + \cos A)}.$$

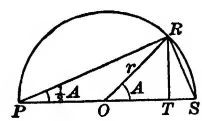


Fig. 145.

(Suggestion. $\sin \frac{1}{2}A = \dfrac{RS}{2r}$, $\overline{RS}^2 = \overline{OR}^2 + \overline{OS}^2 - 2\,OR \cdot OS\cos A$,

$\cos \frac{1}{2}A = \dfrac{RP}{2r}$, $\overline{RP}^2 = \overline{OR}^2 + \overline{OP}^2 + 2\,OR \cdot OP\cos A$.)

17. If $A + B + C = 180°$, show that

$$\tan A + \tan B + \tan C = \tan A\tan B\tan C.$$

18. $\tan A = \dfrac{2\tan\frac{1}{2}A}{1 - \tan^2\frac{1}{2}A}$. Solve this equation for $\tan\frac{1}{2}A$, and identify your result with (4), Art. 112.

19. Show that the equation $a\tan x + b\cot x = c$, may be reduced to the form $(a - b)\cos 2x + c\sin 2x = a + b$.

20. A flagpole 50 ft. high stands on a tower 40 ft. high. At what distance from the foot of the tower will the flagpole and tower subtend equal angles? *Ans.* 120 ft.

21. A tower is situated at a distance of a ft. from the banks of a river b ft. wide. At what height on the tower will the river subtend an angle of 30°?

Ans. $\frac{1}{2} b\sqrt{3} \pm \frac{1}{2}\sqrt{3 b^2 - 4 a^2 - 4 ab}$. (Two solutions?)

22. At a distance of 1000 ft. from the foot of a tower which contains a town clock the dial of the clock subtends an angle of 30′. The center of the dial is 193.5 ft. above the level of the ground. Find the diameter of the dial.

Ans. 9 ft.

23. The height h of an object AB was computed from the distance d of a point O from the foot of the object and the angle θ which AB subtended at this point. It was found that an error ϵ had been made in measuring θ. Show that h must be corrected by an amount

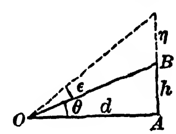


Fig. 146.

$$\eta = \frac{d \sin \epsilon}{\cos(\theta + \epsilon)\cos\theta}.$$

113. Sums and Differences Transformed into Products. From

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$
$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

we obtain by addition and subtraction

$$\sin(A + B) + \sin(A - B) = 2 \sin A \cos B$$
$$\sin(A + B) - \sin(A - B) = 2 \cos A \sin B$$
$$\cos(A + B) + \cos(A - B) = 2 \cos A \cos B$$
$$\cos(A + B) - \cos(A - B) = -2 \sin A \sin B.$$

Let us now put

$$A + B = x \text{ and } A - B = y,$$

from which

$$A = \tfrac{1}{2}(x + y), \quad B = \tfrac{1}{2}(x - y),$$

so that the preceding formulas become

$$\left.\begin{aligned} \sin x + \sin y &= 2 \sin \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y) \\ \sin x - \sin y &= 2 \cos \tfrac{1}{2}(x + y) \sin \tfrac{1}{2}(x - y) \\ \cos x + \cos y &= 2 \cos \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y) \\ \cos x - \cos y &= -2 \sin \tfrac{1}{2}(x + y) \sin \tfrac{1}{2}(x - y). \end{aligned}\right\} \qquad (1)$$

These formulas are frequently used in the further study of mathematics. One of their uses is that they enable us to replace sums or differences by products and thus help us to adapt formulas to computation by logarithms.

EXAMPLE 1. Transform $\sin 2\phi + \sin 4\phi + \sin 6\phi$ into a product.

Solution. By the first formula (1)

$$\begin{aligned} \sin 2\phi + \sin 4\phi &= 2 \sin \tfrac{1}{2}(2\phi + 4\phi) \cos \tfrac{1}{2}(2\phi - 4\phi) \\ &= 2 \sin 3\phi \cos \phi \end{aligned}$$

and by (1), Art. 111,

$$\sin 6\phi = 2 \sin 3\phi \cos 3\phi.$$

Adding

$$\sin 2\phi + \sin 4\phi + \sin 6\phi = 2 \sin 3\phi (\cos 3\phi + \cos \phi).$$

We next apply the third of formula (1) to the expression in parentheses on the right

$$\cos 3\phi + \cos \phi = 2 \cos 2\phi \cos \phi,$$

so that finally

$$\sin 2\phi + \sin 4\phi + \sin 6\phi = 4 \sin 3\phi \cos 2\phi \cos \phi.$$

EXAMPLE 2. If $A + B + C = 180°$, show that

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

Solution.
$$\begin{aligned} \sin 2A + \sin 2B &= 2 \sin (A + B) \cos (A - B) \\ &= 2 \sin C \cos (A - B), \end{aligned}$$

and

$$\begin{aligned} \sin 2C &= 2 \sin C \cos C \\ &= -2 \sin C \cos (A + B); \end{aligned}$$

hence

$$\sin 2A + \sin 2B + \sin 2C = 2 \sin C\,[\cos (A - B) - \cos (A + B)];$$

but by the fourth of the formulas (1)

$$\cos (A - B) - \cos (A + B) = 2 \sin A \sin B,$$

therefore finally

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

Exercise 51

1. State in words the theorems embodied in the formulas (1), Article 113.

2. Show that

$$\sin (30^\circ + y) + \sin (30^\circ - y) = \cos y,$$
$$\sin (30^\circ + y) - \sin (30^\circ - y) = \sqrt{3} \sin y,$$
$$\cos (30^\circ + y) + \cos (30^\circ - y) = \sqrt{3} \cos y,$$
$$\cos (30^\circ + y) - \cos (30^\circ - y) = - \sin y.$$

3. Show that

$$\sin 75^\circ + \sin 15^\circ = \tfrac{1}{2}\sqrt{6},$$
$$\sin 75^\circ - \sin 15^\circ = \tfrac{1}{2}\sqrt{2},$$
$$\cos 75^\circ + \cos 15^\circ = \tfrac{1}{2}\sqrt{6},$$
$$\cos 75^\circ - \cos 15^\circ = -\tfrac{1}{2}\sqrt{2}.$$

4. Express the following products as sums or differences of two functions,

$$\sin 10^\circ \cos 5^\circ,\ \cos 20^\circ \sin 10^\circ,\ \sin \tfrac{1}{2}\theta \sin \tfrac{1}{3}\theta,$$
$$\cos (\tfrac{1}{4}\pi - \theta) \cos (\tfrac{1}{4}\pi + \theta).$$

Ans. $\frac{1}{2}(\sin 15^\circ + \sin 5^\circ)$, ... $\frac{1}{2}(\cos \frac{1}{2}\pi + \cos 2\theta) = \frac{1}{2}\cos 2\theta$.

5. Show that

$$\sin 16^\circ + \sin 14^\circ = 2 \sin 15^\circ \cos 1^\circ,$$
$$\sin \frac{3x}{2} - \sin \frac{x}{2} = 2 \cos x \sin \frac{x}{2},$$
$$\sin (n + 1)x + \sin (n - 1)x = 2 \sin nx \cos x.$$

Prove the following identities:

6. $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \dfrac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}.$

7. $\dfrac{\cos A + \cos B}{\cos A - \cos B} = -\cot \frac{1}{2}(A+B) \cot \frac{1}{2}(A-B).$

8. $\dfrac{\sin A - \sin B}{\cos A + \cos B} = \tan \frac{1}{2}(A-B).$

9. $\dfrac{\sin A - \sin B}{\cos A - \cos B} = -\cot \frac{1}{2}(A+B).$

10. Show that

$$\sin(\alpha + x)\sin x = m\cos(\alpha + x)\cos x$$

may be transformed into

$$-\cos(\alpha + 2x) + \cos\alpha = m[\cos(\alpha + 2x) + \cos\alpha].$$

11. Show that

$$\frac{1}{2}(\cos 2x + \cos 2y) = \cos(x+y)\cos(x-y),$$
$$-\frac{1}{2}(\cos 2x - \cos 2y) = \sin(x+y)\sin(x-y).$$

12. Using the results of 11 show that

$$\cos(x+y)\sin(x-y) + \cos(y+z)\sin(y-z) + \cos(z+x)\sin(z-x) = 0.$$

13. Show that

$$\sin 100^\circ + \sin 40^\circ + \sin 60^\circ = 4\cos 30^\circ \sin 50^\circ \cos 20^\circ,$$
$$\sin 2\theta + \sin 6\theta + \sin 8\theta = 4\cos\theta\cos 3\theta\sin 4\theta.$$

14. Show that

$$\sin x + \sin(x-2) = 2\sin(x-1)\cos 1,$$

whence $\sin x = 2\sin(x-1)\cos 1 - \sin(x-2).$

Similarly $\sin x = 2\cos(x-1)\sin 1 + \sin(x-2),$

$$\cos x = 2\cos(x-1)\cos 1 - \cos(x-2),$$
$$\cos x = -2\sin(x-1)\sin 1 + \cos(x-2).$$

These formulas enable us to compute the sine or cosine of x° if the sines and cosines of $(x-1)^\circ$, $(x-2)^\circ$, and 1° are known.

15. Assuming the functions of 1° and 2° as known, compute the sines and cosines of 3°, 4° and 5°.

(Suggestion. Apply the results of Problem 14.)

16. By the law of sines

$$\frac{a}{b} = \frac{\sin A}{\sin B}.$$

Taking this proportion by composition and division, we obtain

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B}.$$

Apply formulas (1), Art. 113, to this result and deduce the law of tangents. (This constitutes an independent *proof of the law of tangents*. This proof is the one generally given in elementary textbooks on trigonometry.)

17. From the law of sines we obtain readily

$$\frac{c}{a-b} = \frac{\sin C}{\sin A - \sin B}, \qquad \frac{c}{a+b} = \frac{\sin C}{\sin A + \sin B},$$

Hence deduce the *double formulas*

$$\frac{c}{a-b} = \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)}, \qquad \frac{c}{a+b} = \frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)}.$$

18. In Fig. 83, let A' = angle PAC, B' = angle PBC, then by applying the law of sines to each of the triangles PAC and PBC, we find

$$d_3 = \frac{a \sin B'}{\sin \beta} = \frac{b \sin A'}{\sin \alpha}, \tag{1}$$

from which

$$\frac{\sin A'}{\sin B'} = \frac{a \sin \alpha}{b \sin \beta}.$$

By composition and division and the identity in Problem 6

$$\frac{\sin A' + \sin B'}{\sin A' - \sin B'} = \frac{a \sin \alpha + b \sin \beta}{a \sin \alpha - b \sin \beta} = \frac{\tan \frac{1}{2}(A' + B')}{\tan \frac{1}{2}(A' - B')}. \tag{2}$$

Now $A' + B'$ is known from the relation

$$A' + B' + C + \alpha + \beta = 360°,$$

and therefore (2) enables us to find $A' - B'$. Hence A' and B' may be found and with these known, the law of sines gives

$$d_1 = \frac{b \sin (A' + \alpha)}{\sin \alpha}, \qquad d_2 = \frac{a \sin (B' + \beta)}{\sin \beta},$$

This constitutes another solution of the *three-point problem.*

19. Prove the formulas (1), Article 113, geometrically by means of Fig. 147.

Suggestion. Take the radius of the arc equal to unity, that is, $OE = OD = 1$, then

$$\sin x + \sin y = 2 \cdot GF,$$
$$\sin x - \sin y = 2 \cdot LF,$$
$$\cos x + \cos y = 2 \cdot OG,$$
$$\cos x - \cos y = -2 \cdot LD.$$

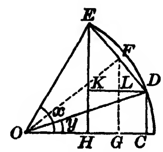


Fig. 147.

20. Show that

$$\frac{\tan(\alpha + x) + \tan x}{\tan(\alpha + x) - \tan x} = \frac{\sin(\alpha + 2x)}{\sin \alpha}.$$

CHAPTER XII

TRIGONOMETRIC EQUATIONS

114. Angles Corresponding to a Given Function. Every given angle has a single sine, cosine, tangent, etc. In Chapter X it was shown how these functions may be found from the tables by first expressing them in terms of an angle in the first quadrant. Suppose now that one of the functions of an angle is known and it is required to find the angle. If the angle is known to be less than R, it is at once found from the tables; if less than $2R$, there also is no uncertainty except in the case of the sine (and its reciprocal the cosecant), which may be either the angle given in the table or its supplement. But in case no restriction is imposed on the magnitude of the angle, the given function may belong to any one of an unlimited number of angles. We shall show how all the angles corresponding to a given function can in each case be expressed by a single formula from which their separate values may be written down when required.

115. Principal Value. Of all the angles which correspond to a given function, the one which has the least numerical value is called the principal value (see Article 81). If there are two least values with opposite signs, the positive angle is taken as the principal value.

Thus, if $\sin\theta = \frac{1}{2}$, $\theta = 30°$ or $150°$ or either of these values increased or diminished by any number of times $360°$. Of all these angles, $30°$ being the least numerically, is considered the principal value of the angles whose sine is $\frac{1}{2}$.

If $\sin\theta = -\frac{1}{2}\sqrt{3}$, $\theta = -60°$, $-120°$ or either of these values increased or diminished by any number of times $360°$. Here $-60°$, having the least numerical value, is the principal value of the angles whose sine is $-\frac{1}{2}\sqrt{3}$.

If $\cos\theta = \frac{1}{2}\sqrt{2}$, $\theta = 45°$ or $-45°$ or either of these increased or diminished by multiples of $360°$. In this case $45°$, not $-45°$, is the principal value of θ.

116. Formula for Angles Having a Given Sine.

Let it be required to determine θ from the equation

$$\sin\theta = k.$$

Let α = the principal value of θ. Then the other values of θ are

$$\begin{array}{ll} \pi - \alpha, & -\pi - \alpha, \\ 2\pi + \alpha, & -2\pi + \alpha, \\ 3\pi - \alpha, & -3\pi - \alpha, \\ 4\pi + \alpha, & -4\pi + \alpha, \\ 5\pi - \alpha, & -5\pi - \alpha, \\ 6\pi + \alpha & -6\pi + \alpha \\ \text{etc.} & \text{etc.} \end{array}$$

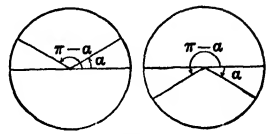


Fig. 148.

It will be observed that the coefficients of π are the integers

1, 2, 3, 4, 5, 6, etc., − 1, − 2, − 3, − 4, − 5, − 6, etc.,

while the sign of α is positive or negative according as the coefficient of π is even or odd. Now $(-1)^n$ is always positive when n is an even integer, negative when n is an odd integer. Making use of this property, we can express the whole set of angles by the single formula

$$\theta = n\pi + (-1)^n\alpha,$$

where α is the principal value of the angles whose sine is k, and n any positive or negative integer.

When $n = 0$, we get the principal value of θ, for

$$0\pi + (-1)^0\alpha = \alpha.$$

117. Formula for Angles Having a Given Cosine.

If $\cos\theta = k$, and α the principal value of θ, the other values of θ are

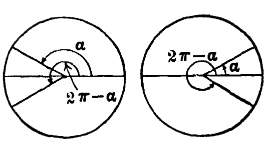


Fig. 149.

$$\begin{array}{ll} 2\pi - \alpha, & -2\pi - \alpha, \\ 2\pi + \alpha, & -2\pi + \alpha, \\ 4\pi - \alpha, & -4\pi - \alpha, \\ 4\pi + \alpha, & -4\pi + \alpha, \\ 6\pi - \alpha, & -6\pi - \alpha, \\ 6\pi + \alpha, & -6\pi + \alpha, \\ \text{etc.} & \text{etc.} \end{array}$$

All these values and no others are expressed by the single formula

$$\theta = 2\,n\pi \pm \alpha,$$

where $2\,n$ is necessarily an even number.

118. Formula for Angles Having a Given Tangent.

If $\tan\theta = k$, and α the principal value of θ, the other values of θ are

$$\begin{array}{ll} \pi + \alpha, & -\pi + \alpha, \\ 2\pi + \alpha, & -2\pi + \alpha, \\ 3\pi + \alpha, & -3\pi + \alpha, \\ 4\pi + \alpha, & -4\pi + \alpha, \\ 5\pi + \alpha, & -5\pi + \alpha, \\ \text{etc.} & \text{etc.} \end{array}$$

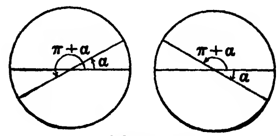


Fig. 150.

All these values and no others are expressed by the single formula

$$\theta = n\pi + \alpha,$$

n being any positive or negative integer.

119. Summary of Results. Summing up the results of the last three articles we have

$$\left.\begin{array}{ll} \sin(n\pi + (-1)^n\alpha) & = \sin\alpha, \\ \cos(2n\pi \pm \alpha) & = \cos\alpha, \\ \tan(n\pi + \alpha) & = \tan\alpha, \end{array}\right\} \qquad (1)$$

where α is the principal value of the angle, and n any positive or negative integer.

These three formulas should be thoroughly understood and memorized by the student.

120. Trigonometric Equations Involving a Single Angle.

Equations involving one or more trigonometric functions of one or more unknown angles are called trigonometric equations. They may or may not involve other unknowns which are not angles.

To solve trigonometric equations involving a single unknown angle, we express each of the functions which occur in the equation in terms of some one of them, and solve the resulting equation alge-

braically, considering this function as the unknown. The corresponding angles may then be easily found from the formulas of Article 119.

EXAMPLE 1. Solve the equation $2 \sin x + \csc x = 3$.

Solution. Expressing the cosecant in terms of the sine, we have

$$2 \sin x + \frac{1}{\sin x} = 3.$$

Now consider $\sin x$ the unknown, call it s say, we then have

$$2s + \frac{1}{s} = 3.$$

Clearing of fractions and transposing,

$$2s^2 - 3s - 1 = 0.$$

Solving $$s = \frac{3 \pm \sqrt{3^2 - 4 \cdot 2}}{4} = \frac{3 \pm 1}{4} = 1 \quad \text{or} \quad \tfrac{1}{2},$$

that is, $\sin x = 1$ or $\frac{1}{2}$,

from which the principal values of x are $x = \frac{\pi}{2}$ or $\frac{\pi}{6}$,

and by Article 119 the general values of x are

$$x = n\pi + (-1)^n \frac{\pi}{2}, \quad \text{or} \quad n\pi + (-1)^n \frac{\pi}{6}.$$

The general values of x just written are equivalent to the two sets of values

$$x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}, \text{etc.}, \frac{-3\pi}{2}, \frac{-7\pi}{2}, \frac{-11\pi}{2}, \text{etc.},$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \text{etc.}, \frac{-7\pi}{6}, \frac{-11\pi}{6}, \frac{-19\pi}{6}, \text{etc.},$$

Check. If $\sin x = 1$, $\csc x = 1$, and $2 \sin x + \csc x = 2 \cdot 1 + 1 = 3$.
If $\sin x = \frac{1}{2}$, $\csc x = 2$, and $2 \sin x + \csc x = 2 \cdot \frac{1}{2} + 2 = 3$.

EXAMPLE 2. To find θ in the equation $\tan \theta + 3 \cot \theta = 4$.

Solution. Expressing the cotangent in terms of the tangent gives

$$t + \frac{3}{t} = 4,$$

where $t = \tan\theta$. Solving for t,

$$t^2 - 4t + 3 = 0,$$

$$t = \frac{4 \pm \sqrt{4^2 - 4\cdot 3}}{2} = \frac{4 \pm 2}{2} = 3 \text{ or } 1,$$

that is, $\quad \tan\theta = 3 \quad$ or $\quad 1,$

from which the principal values of θ are

$$\theta = \tan^{-1} 3 = 71^\circ\, 33'\, 54'' \text{ or } \tan^{-1} 1 = 45^\circ.$$

By Article 119 the general values of θ are

$$\theta = n \cdot 180^\circ + 71^\circ\, 33'\, 54'' \text{ or } n \cdot 180^\circ + 45^\circ.$$

Check. If $\tan\theta = 3$, $\cot\theta = \frac{1}{3}$, $\quad \tan\theta + 3\cot\theta = 3 + 3 \cdot \frac{1}{3} = 4.$
If $\tan\theta = 1$, $\cot\theta = 1$, $\quad \tan\theta + 3\cot\theta = 1 + 3 \cdot 1 = 4.$

EXAMPLE 3. Find ϕ in $\tan\phi + 2\sqrt{3}\cos\phi = 0$.

Solution. $$\tan\phi = \frac{\sqrt{1 - \cos^2\phi}}{\cos\phi},$$

hence $$\sqrt{1 - \cos^2\phi} + 2\sqrt{3}\cos^2\phi = 0.$$

Transposing one term and squaring,

$$1 - \cos^2\phi = 12\cos^4\phi,$$

$$12\cos^4\phi + \cos^2\phi - 1 = 0,$$

$$\cos^2\phi = \frac{-1 \pm \sqrt{1 + 4 \cdot 12}}{24} = \frac{-1 \pm 7}{24} = \frac{1}{4} \text{ or } \frac{-1}{3},$$

$$\cos\phi = \tfrac{1}{2}, \quad -\tfrac{1}{2}, \quad \sqrt{-\tfrac{1}{3}}, \quad -\sqrt{-\tfrac{1}{3}}.$$

The last two values of $\cos\phi$ are not admissible, since the cosine of a real angle cannot be imaginary, and the first two values each give

$$\phi = 2n\pi \pm \frac{\pi}{3}.$$

Check. $\cos\left(2n\pi \pm \dfrac{\pi}{3}\right) = \frac{1}{2}$, $\tan\left(2n\pi \pm \dfrac{\pi}{3}\right) = \pm\sqrt{3}$,

$\tan\phi + 2\sqrt{3}\cos\phi = \pm 3 + 2\sqrt{3}\cdot(\frac{1}{2}) = 0$ for the lower sign but not for the upper. Hence the general solution of the equation

$$\tan\phi + 2\sqrt{3}\cos\phi = 0 \quad \text{is} \quad \phi = 2n\pi - \frac{\pi}{3}.$$

This example clearly shows the necessity of verifying the results before accepting them as the solutions of a given equation.

EXAMPLE 4. Solve the equation $\sin x + \cos x = \sqrt{2}$.

First solution. $\sin x + \sqrt{1 - \sin^2 x} = \sqrt{2}$,

$$\sqrt{1 - \sin^2 x} = \sqrt{2} - \sin x.$$

Squaring $\quad 1 - \sin^2 x = 2 - 2\sqrt{2}\sin x + \sin^2 x.$

Solving $\quad 2\sin^2 x - 2\sqrt{2}\sin x + 1 = 0,$

$$\sin x = \frac{2\sqrt{2} \pm \sqrt{8 - 4 \cdot 2}}{4} = \frac{\sqrt{2}}{2},$$

$$x = n\pi + (-1)^n \frac{\pi}{4}.$$

Check. $\sin\left(n\pi + (-1)^n \frac{\pi}{4}\right) = \frac{1}{2}\sqrt{2}$, $\cos\left(n\pi + (-1)^n \frac{\pi}{4}\right) = \pm \frac{1}{2}\sqrt{2}$, according as n is even or odd. Therefore

$$\sin x + \cos x = \tfrac{1}{2}\sqrt{2} \pm \tfrac{1}{2}\sqrt{2} = \sqrt{2} \quad \text{or} \quad 0$$

according as n is even or odd. We see that the lower sign does not satisfy the original equation, that is n cannot be odd; hence the general solution of the equation

$$\sin x + \cos x = \sqrt{2}$$

is $\quad x = 2n\pi + \frac{1}{4}\pi.$

Second solution. We have given the solution which would most naturally suggest itself to the beginner. A more elegant solution of the foregoing equation is the following:

Since $\cos x = \sin\left(\frac{1}{2}\pi - x\right)$, the equation to be solved may be written

$$\sin x + \cos x = \sin x + \sin\left(\tfrac{1}{2}\pi - x\right) = \sqrt{2}.$$

The middle member is the sum of two sines, which by Article 113 may be transformed into a product, thus

$$\sin x + \sin\left(\tfrac{1}{2}\pi - x\right) = 2\sin\tfrac{1}{4}\pi\cos\left(x - \tfrac{1}{4}\pi\right) = \sqrt{2},$$

that is $\quad 2 \cdot \frac{1}{2}\sqrt{2} \cdot \cos\left(x - \frac{1}{4}\pi\right) = \sqrt{2},$

or $\quad \cos\left(x - \frac{1}{4}\pi\right) = 1,$

from which $\quad x - \frac{1}{4}\pi = 2n\pi \pm 0$, or $x = 2n\pi + \frac{1}{4}\pi.$

EXAMPLE 5. Solve the equation $a \sin(x+\alpha) + b\cos(x+\beta) = 0$, a, b, α, β, being known constants.

Solution. $\sin(x+\alpha) = \sin x \cos\alpha + \cos x \sin\alpha$,

$\cos(x+\beta) = \cos x \cos\beta - \sin x \sin\beta$.

Substituting these values in the original equation,

$$a(\sin x \cos\alpha + \cos x \sin\alpha) + b(\cos x \cos\beta - \sin x \sin\beta) = 0.$$

Collecting the coefficients of $\sin x$ and $\cos x$ separately, we have

$$(a\cos\alpha - b\sin\beta)\sin x + (a\sin\alpha + b\cos\beta)\cos x = 0.$$

Dividing by $\cos x$, and solving for $\tan x$,

$$\tan x = -\frac{a\sin\alpha + b\cos\beta}{a\cos\alpha - b\sin\beta},$$

from which

$$x = n\pi - \tan^{-1}\left(\frac{a\sin\alpha + b\cos\beta}{a\cos\alpha - b\sin\beta}\right).$$

Check. To check complicated results like the one just obtained it is best to assume arbitrary values for the constants. Thus if

$$a = 2,\ b = 1,\ \alpha = 20^\circ,\ \beta = 15^\circ,\ \frac{a\sin\alpha + b\cos\beta}{a\cos\alpha - b\sin\beta} = 1.0181,$$

$$x = n\pi - \tan^{-1} 1.0181 = n\pi - 45^\circ 31',$$

$$x + \alpha = n\pi - 25^\circ 31',\quad x + \beta = n\pi - 30^\circ 31'.$$

Substituting these values in the original equation, we have

$$2(\pm 0.4308) + 1(\mp 0.8615) = 0.$$

EXERCISE 52

In the following find both the principal and the general value of the angle:

1. $\tan\theta = 2\sin\theta$.

Ans. $\theta = n\pi$ or $2n\pi \pm \frac{\pi}{3}$; principal values 0 and $\frac{\pi}{3}$.

2. $3\sin^2 x = \cos^2 x$.

Ans. $x = n\pi \pm \frac{\pi}{6}$; principal value $\frac{\pi}{6}$.

3. $3\tan^2 x - 4\sin^2 x = 1$.

Ans. $x = n\pi \pm \frac{\pi}{4}$; principal value $\frac{\pi}{4}$.

4. $2\sin^2\phi = 3\cos\phi$.

Ans. $\phi = 2n\pi \pm \frac{\pi}{3}$; principal value $\frac{\pi}{3}$.

5. $\tan y + \cot y = 2$. *Ans.* $y = \pi n + \frac{1}{4}\pi$; principal value $\frac{\pi}{4}$.

In the following problems find those values of the variable angle which are less than 360°:

6. $\tan^2 x + \csc^2 x = 3$. *Ans.* $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

7. $\sin\theta + \cos\theta = 1$. *Ans.* $\theta = 0, \frac{1}{2}\pi$.

8. $\csc x \cot x = 2\sqrt{3}$. *Ans.* $x = \frac{\pi}{6}, \frac{11\pi}{6}$.

9. $\sin^2 t - 2\cos t + \frac{1}{4} = 0$. *Ans.* $t = \frac{\pi}{3}, \frac{5\pi}{3}$.

10. $3\sec^4\theta - 10\sec^2\theta + 8 = 0$.

Ans. $\theta = \frac{\pi}{6}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{6}$.

11. $\tan x + \sec^2 x = 7$.

Ans. $x = 63°\,26'\,04''$, $108°26'\,06''$, $243°\,26'\,04''$, $288°\,26'\,06''$.

12. $6\cos^2 x + 5\sin x = 7$.

Ans. $x = 19°\,28'\,16''$, $30°$, $150°$, $160°\,31'\,44''$.

13. $\sin x + \csc x = \frac{5}{2}$. *Ans.* $x = 30°, 150°$.

14. $\sin x - \cos x = \frac{5}{2}$. *Ans.* No solution.

15. If $a\sin x + b\cos x = c$, show that $\sin x = \dfrac{ac \pm b\sqrt{a^2+b^2-c^2}}{a^2+b^2}$.

16. If $a\tan x + b\cot x = c$, show that $\tan x = \dfrac{c \pm \sqrt{c^2 - 4ab}}{2a}$.

17. If $\sin(\alpha + x) = m\sin x$, show that $\sin x = \dfrac{\pm\sin\alpha}{\sqrt{m^2 - 2m\cos\alpha + 1}}$,

or more simply $\cot x = \dfrac{m - \cos\alpha}{\sin\alpha}$.

18. If $\tan(\alpha + x) = m\tan x$, show that

$$\tan x = \frac{(m-1) \pm \sqrt{(m-1)^2 - 4m\tan^2\alpha}}{2m\tan\alpha}.$$

19. If $\tan(\alpha + x)\tan x = m$, show that

$$\tan x = \frac{-(1+m)\tan\alpha \pm \sqrt{(1+m)^2\tan^2\alpha + 4m}}{2}.$$

By expressing each product as a sum or difference of functions show:

20. If $\sin(\alpha \pm x)\sin x = m$, then $\cos(\alpha \pm 2x) = \cos\alpha \mp 2m$.
21. If $\cos(\alpha \pm x)\cos x = m$, then $\cos(\alpha \pm 2x) = 2m - \cos\alpha$.
22. If $\sin(\alpha \pm x)\cos x = m$, then $\sin(\alpha \pm 2x) = 2m - \sin\alpha$.
23. If $\cos(\alpha \pm x)\sin x = m$, then $\sin(\alpha \pm 2x) = \sin\alpha \pm 2m$.

121. Trigonometric Equations Involving Multiple Angles.

When an equation involves multiple angles, as the equation

$$\cos 3\theta + \sin 2\theta = a, \quad \text{or} \quad \tan 2x = \cot 5x,$$

it can frequently be solved by two or more different methods, and the answer will appear in various forms according as one or the other of these methods has been employed in the solution. Generally the different forms of the answers can easily be identified, but in some cases considerable ingenuity is required to show that the different forms are really the same. Thus it is easy to see that $2n\pi \pm \frac{1}{2}\pi$ and $n\pi + \frac{1}{2}\pi$ (n being any integer, but not the same integer in both forms) express the same general value, but it is not so easy to see that

$$\theta = n\pi + \tfrac{1}{2}\pi \quad \text{or} \quad n\pi + (-1)^n \sin^{-1}\tfrac{1}{4}(-1 \pm \sqrt{5})$$

$$\text{and } \theta = 2n\pi - \tfrac{1}{2}\pi \quad \text{or} \quad \frac{2n\pi}{5} + \frac{\pi}{10}$$

are equivalent results.

EXAMPLE 1. Solve the equation $\sin 2\theta = \cos\theta$.

First solution. Substituting for $\sin 2\theta$ its value in terms of the single angle θ (Article 111, (1)), we have

$$2\sin\theta\cos\theta = \cos\theta.$$

Transposing and factoring,

$$\cos\theta(2\sin\theta - 1) = 0,$$

from which

$$\cos\theta = 0, \quad \text{or} \quad \sin\theta = \tfrac{1}{2}.$$

Therefore

$$\theta = 2n\pi \pm \tfrac{1}{2}\pi = n\pi + \tfrac{1}{2}\pi^*, \text{ or } n\pi + (-1)^n\frac{\pi}{6}. \qquad (1)$$

* $\theta = 2n\pi + \frac{1}{2}\pi$ and $2n\pi - \frac{1}{2}\pi = (2n-1)\pi + \frac{1}{2}\pi$. Now $2n$ represents every even and $2n - 1$ every odd number, hence we may write $\theta = n\pi + \frac{1}{2}\pi$, where n is any integer, even or odd.

Check.

$$\sin 2\theta = \sin(2n\pi + \pi) = 0, \text{ or } \sin\left(2n\pi + (-1)^n\frac{\pi}{3}\right) = \pm\tfrac{1}{2}\sqrt{3},$$

according as n is even or odd.

$$\cos\theta = \cos(n\pi + \tfrac{1}{2}\pi) = 0, \text{ or } \cos\left(n\pi + (-1)^n\frac{\pi}{6}\right) = \pm\tfrac{1}{2}\sqrt{3},$$

according as n is even or odd.

Substituting these values in the original equation, we have

$$0 = 0, \text{ or } \pm\tfrac{1}{2}\sqrt{3} = \pm\tfrac{1}{2}\sqrt{3}.$$

Second solution.

Since $\cos\theta = \sin(\tfrac{1}{2}\pi - \theta)$, we have

$$\sin 2\theta = \sin(\tfrac{1}{2}\pi - \theta),$$

from which

$$2\theta = n\pi + (-1)^n(\tfrac{1}{2}\pi - \theta),$$

or

$$\theta = \frac{n\pi + (-1)^n\tfrac{1}{2}\pi}{2 + (-1)^n}. \tag{2}$$

Third solution. Transposing the second member of the equation, we have

$$\sin 2\theta - \cos\theta = \sin 2\theta - \sin(\tfrac{1}{2}\pi - \theta) = 0.$$

By Article 113, the difference of two sines may be tranformed into a product of a sine and cosine, thus

$$\sin 2\theta - \sin(\tfrac{1}{2}\pi - \theta) = 2\cos(\tfrac{1}{2}\theta + \tfrac{1}{4}\pi)\sin(\tfrac{3}{2}\theta - \tfrac{1}{4}\pi) = 0,$$

from which

$$\cos(\tfrac{1}{2}\theta + \tfrac{1}{4}\pi) = 0, \text{ or } \sin(\tfrac{3}{2}\theta - \tfrac{1}{4}\pi) = 0,$$

and

$$\tfrac{1}{2}\theta + \tfrac{1}{4}\pi = 2n\pi \pm \tfrac{1}{2}\pi = n\pi + \tfrac{1}{2}\pi, \text{ or } \tfrac{3}{2}\theta - \tfrac{1}{4}\pi = n\pi,$$

that is,

$$\theta = 2n\pi + \tfrac{1}{2}\pi, \text{ or } \tfrac{2}{3}n\pi + \tfrac{1}{6}\pi. \tag{3}$$

Identification of results.

If in (2) n is odd, say $2m + 1$, we have

$$\theta = (2m + 1)\pi - \tfrac{1}{2}\pi = 2m\pi + \tfrac{1}{2}\pi,$$

and if n is even, say $2m$, we have

$$\theta = \frac{2m\pi + \tfrac{1}{2}\pi}{3} = \tfrac{2}{3}m\pi + \tfrac{1}{6}\pi.$$

This shows that the results (2) and (3) are the same.

Next consider the second value in (3). Every integer n is either some multiple of 3, say $3m$, or some multiple of 3 increased by unity, say $3m+1$, or some multiple of 3 increased by 2, say $3m+2$.

In the first case, $n = 3m$, we have from the second value of (3)

$$\frac{2n\pi}{3}+\frac{\pi}{6}=\frac{2.3\,m\pi}{3}+\frac{\pi}{6}=2m\pi+\frac{\pi}{6}.$$

In the second case, $n = 3m+1$,

$$\frac{2n\pi}{3}+\frac{\pi}{6}=\frac{2(3m+1)\pi}{3}+\frac{\pi}{6}=(2m+1)\pi-\frac{\pi}{6}.$$

In the third case, $n = 3m+2$,

$$\frac{2n\pi}{3}+\frac{\pi}{6}=\frac{2(3m+2)\pi}{3}+\frac{\pi}{6}=(2m+1)\pi+\frac{\pi}{2}.$$

This last value, combined with the first value of θ in (3), gives

$\theta=2n\pi+\frac{1}{2}\pi$, or $(2m+1)\pi+\frac{1}{2}\pi$, that is, $\theta=n\pi+\frac{1}{2}\pi$ (n even or odd),

and

$\theta = 2m\pi+\frac{1}{6}\pi$, or $(2m+1)\pi-\frac{1}{6}\pi$, may be written

$$\theta = n\pi+(-1)^n\frac{\pi}{6}.$$

This shows that the results (3) and (1) are the same.

Exercise 53

Find the general value of the angle in each of the following equations:

1. $\cos 2x+\cos x+1=0$. *Ans.* $x=n\pi-\frac{1}{2}\pi$, or $2n\pi\pm\frac{2}{3}\pi$.
2. $\cos 5x=\sin 4x$. *Ans.* $x=2n\pi-\frac{1}{2}\pi$, or $\dfrac{2n\pi}{9}+\dfrac{\pi}{18}$.
3. $\cos 5x=\cos 4x$. *Ans.* $x=2n\pi$, or $\dfrac{2n\pi}{9}$.
4. $\sin 4x=\sin 5x$. *Ans.* $x=2n\pi$, or $(2n+1)\dfrac{\pi}{9}$.
5. $\tan 5\theta=\cot 2\theta$. *Ans.* $\theta=\dfrac{n\pi}{7}+\dfrac{\pi}{14}$.
6. $\sin 3x+\sin 2x+\sin x=0$.

   Ans. $x=\dfrac{n\pi}{2}$, or $2n\pi\pm\dfrac{2\pi}{3}$.

7. $\cos x - \cos 3x = \sin 2x$.

Ans. $x = n\pi$, or $n\pi - \frac{1}{2}\pi$, or $n\pi + (-1)^n \frac{\pi}{6}$.

8. $\cos 5y - \cos 3y + \sin y = 0$.

Ans. $y = n\pi$, or $\frac{1}{4}n\pi + (-1)^n \frac{\pi}{24}$.

9. $\cos (60° - x) + \cos (60° + x) = \frac{1}{3}$.

Ans. $x = 2n\pi \pm \cos^{-1}\frac{1}{3} = 2n\,180° \pm 70°\,32'$.

10. $\cos mx = \sin kx$, m and k being known.

Ans. $x = \dfrac{(n\pi + \frac{1}{2}\pi)}{(m \pm k)}$.

11. $\cot \phi = \tan k\phi$, k being known.

Ans. $\phi = \dfrac{n\pi}{k+1} + \dfrac{\pi}{2(k+1)}$.

12. $\tan 2x \tan x = 1$. *Ans.* $x = n\pi \pm \frac{\pi}{6}$.

Solve each of the following problems by each of three methods, and identify the results.

13. $\cos 2\theta = \sin \theta$. *Ans.* $\theta = 2n\pi - \frac{1}{2}\pi$, or $\frac{2}{3}n\pi + \frac{\pi}{6}$.

14. $\cos 3x = \sin 2x$. *Ans.* $x = 2n\pi - \frac{1}{2}\pi$, or $\frac{2}{5}n\pi + \frac{\pi}{10}$.

Suggestion. $\frac{1}{4}(-1 + \sqrt{5}) = \sin \frac{\pi}{10}$, $\frac{1}{4}(-1 - \sqrt{5}) = \sin\left(-\frac{3\pi}{10}\right)$.

15. $\sin 3x = \cos 2x$. *Ans.* $x = 2n\pi + \frac{1}{2}\pi$, or $\frac{2}{5}n\pi + \frac{\pi}{10}$.

122. Trigonometric Equations Involving Two or More Variables. When there are given two or more trigonometric equations, involving two or more variables, the solution, if it is possible at all, generally depends upon more or less ingenious combinations of the equations, for which no definite rules can be given. We shall illustrate the various methods and devices commonly employed by a few examples chosen from among those which most frequently occur in applied trigonometry.

EXAMPLE 1. To find r and θ from the equations

$$r \sin \theta = a, \tag{1}$$

$$r \cos \theta = b, \tag{2}$$

a and b being known constants.

First solution. Dividing the first equation by the second, we have

$$\tan \theta = \frac{a}{b}$$

from which θ may be found. r is then found from either (1) or (2),

$$r = \frac{a}{\sin \theta} = \frac{b}{\cos \theta}.$$

Since the angles θ as determined from the tangent differ by multiples of π, $\sin \theta$ and $\cos \theta$ will have two values each which are numerically equal but opposite in sign. Therefore r will have two values which are equal and opposite in sign. If from the outset it is known that r can be positive only, which is the case frequently, then $\sin \theta$ must have the same sign as a and $\cos \theta$ the same sign as b. θ must then be limited to the quadrants determined by these signs.

Second solution. Squaring each equation and adding the results gives

$$r^2 = a^2 + b^2,$$

from which r is obtained. Then θ is found from either of the equations

$$\sin \theta = \frac{a}{r}, \quad \cos \theta = \frac{b}{r}$$

Of the two methods the first is better adapted to the use of logarithms than the second.

EXAMPLE 2. Find r, θ and ϕ from the equations

$$r \cos \theta \cos \phi = a \tag{1}$$

$$r \cos \theta \sin \phi = b \tag{2}$$

$$r \sin \theta = c. \tag{3}$$

First solution. Dividing the second equation by the first we obtain

$$\tan \phi = \frac{b}{a}$$

from which ϕ is obtained. ϕ being known, $r \cos \theta$ is obtained from either equation (1) or (2)

$$r \cos \theta = \frac{a}{\cos \phi} = \frac{b}{\sin \phi}. \tag{4}$$

From (3) and (4) r and θ may be found as in Example 1.

Second solution. Squaring each of the equations (1), (2) and (3) and adding the results gives

$$r^2 = a^2 + b^2 + c^2$$

from which r may be found.

r being known, θ is found from the equation (3), and, r and θ being known, ϕ is found from either equation (1) or (2).

The first solution is preferable if logarithms are to be used throughout.

EXAMPLE 3. Solve the equations

$$r \sin (\alpha + \theta) = a$$
$$r \sin (\beta + \theta) = b$$

for r and θ.

Solution. Applying the addition theorem for the sine, we have

$$r \sin \alpha \cos \theta + r \cos \alpha \sin \theta = a$$
$$r \sin \beta \cos \theta + r \cos \beta \sin \theta = b.$$

Put $r \cos \theta = x$, $r \sin \theta = y$, the equations then become

$$x \sin \alpha + y \cos \alpha = a$$
$$x \sin \beta + y \cos \beta = b.$$

Solving these equations for x and y, we obtain

$$x = r \cos \theta = \frac{a \cos \beta - b \cos \alpha}{\sin \alpha \cos \beta - \sin \alpha \sin \beta} = \frac{a \cos \beta - b \cos \alpha}{\sin (\alpha - \beta)}.$$

$$y = r \sin \theta = \frac{a \sin \beta - b \sin \alpha}{\cos \alpha \sin \beta - \sin \alpha \cos \beta} = \frac{b \sin \alpha - a \sin \beta}{\sin (\alpha - \beta)}.$$

x and y, that is, $r \cos \theta$ and $r \sin \theta$ being known, r and θ are found as in Example 1.

EXERCISE 54

In the following equations, consider r positive.

1. Given $r \sin \theta = 8.219$, $r \cos \theta = 12.88$, find r and θ.
   Ans. $r = 15.28$, $\theta = 32° 33'$.
2. Given $r \sin \theta = 3$, $r \cos \theta = 4$, find r and θ.

3. Given $r \sin \theta = 27.138$, $r \cos \theta = -92.692$, find r and θ.

Ans. $r = 96.583$, $\theta = 163° 40' 52''$.

4. Given
$$\left.\begin{aligned} r\cos\theta\cos\phi &= 59.953,\\ r\cos\theta\sin\phi &= 197.207,\\ r\sin\phi &= 39.062.\end{aligned}\right\}\quad \text{Find } r,\ \theta \text{ and } \phi.$$

Ans. $r = 208.16$, $\theta = 10° 49' 00''$, $\phi = 74° 42' 00''$.

5. Given
$$\left.\begin{aligned} r\sin\theta\cos\phi &= 5,\\ r\cos\theta\sin\phi &= 12,\\ r\sin\theta &= 84.\end{aligned}\right\}\quad \text{Find } r,\ \theta \text{ and } \phi.$$

Ans. By natural functions, $r = 85$, $\theta = 81° 12'$, $\phi = 86° 36'$.

6. Find r, λ and μ from the equations
$$r\cos\lambda\cos\mu = 4,\quad r\cos\lambda\sin\mu = 5,\quad r\sin\lambda = \sqrt{59}.$$

7. $r\sin\left(\frac{\pi}{3} + x\right) = \sqrt{3}$, $r\sin\left(\frac{\pi}{6} + x\right) = 1$. Find r and x.

Ans. $r = 2$, $x = 0°$.

8. Show how to solve the equations
$$r\cos(x - \alpha) = a,\ r\sin(x + \beta) = b, \text{ for } r \text{ and } x.$$

9. Solve the equations
$$\cos(x - y) = \tfrac{1}{2}\sqrt{2},\ \sin(x + y) = \tfrac{1}{2}\sqrt{3},$$
giving both the principal and general values.

Ans. Principal values, $x = \frac{7\pi}{24}$ or $\frac{\pi}{24}$, $y = \frac{\pi}{24}$ or $\frac{7\pi}{24}$.

General values, $x = (m + 2n)\frac{\pi}{2} + (-1)^m\frac{\pi}{6} \pm \frac{\pi}{8}$,

$y = (m - 2n)\frac{\pi}{2} + (-1)^m\frac{\pi}{6} \mp \frac{\pi}{8}$,

where m and n are any two integers.

10. Solve the equations
$$\cos 2x - \cos 2y = a, \qquad \cos x - \cos y = b.$$

Ans. $x = 2n\pi \pm \cos^{-1}\left(\frac{a + 2b^2}{4b^2}\right)$, $y = 2n\pi \pm \cos^{-1}\left(\frac{a - 2b^2}{4b^2}\right)$.

11. Show how to solve the equations
$$\tan(x + y) = a, \qquad \tan x \cdot \tan y = b.$$

123. Solutions Adapted to Logarithmic Computation. The solution of a problem in trigonometry is not considered completed until it can be effected by the use of logarithms, in fact the adaptation of formulas to the use of logarithms forms an important part in trigonometric investigations. Many trigonometric equations whose algebraic solution is exceedingly simple require further treatment from the trigonometric point of view. Equations 15 to 19, Exercise 52, are typical equations of this kind. We will show now how each of these equations may be solved by logarithms.

EXAMPLE 1. To solve the equation

$$a \sin x + b \cos x = c. \qquad (1)$$

Solution. Divide each term of the equation by $\sqrt{a^2+b^2}$, then the coefficients $\dfrac{a}{\sqrt{a^2+b^2}}$ and $\dfrac{b}{\sqrt{a^2+b^2}}$ are fractions the sum of whose squares equals 1, we may therefore put

$$\frac{a}{\sqrt{a^2+b^2}} = \cos\phi, \qquad \frac{b}{\sqrt{a^2+b^2}} = \sin\phi, \qquad (2)$$

and the given equation becomes

$$\sin x \cos\phi + \cos x \sin\phi = \frac{c}{\sqrt{a^2+b^2}},$$

or

$$\sin(x+\phi) = \frac{c}{\sqrt{a^2+b^2}} = \frac{c}{a}\cos\phi = \frac{c}{b}\sin\phi, \qquad (3)$$

and from (2)

$$\tan\phi = \frac{b}{a}. \qquad (4)$$

Having found ϕ from (4), x is found from (3).

An angle like the angle ϕ, introduced to facilitate the solution of a problem, is called an *auxiliary* angle.

NUMERICAL ILLUSTRATION

Suppose the given equation is

$$3.4537 \sin x - 0.9328 \cos x = -1.3794,$$

then $\qquad a = 3.4537, \quad b = -0.9328, \quad c = -1.3794.$

Solution.

By (4)	By (3)
$\log b = 9.96979\,n$ *	$\log c = 0.13969\,n$
$\log a = 0.53828$	$\operatorname{colog} a = 9.46172$
$\log \tan\phi = 9.43151\,n$	$\log\cos\phi = 9.98471$
ϕ† $= -15°\,06'\,52''$,	$\log\sin(x+\phi) = 9.58612\,n$

$$x + \phi = n\pi + (-1)^n\ (22°\,40'\,48'')$$

$$x = n\pi + 15°\,06'\,52'' + (-1)^{n+1}\ (22°\,40'\,48'')$$

The two smallest positive values, ($n = 1, 2$), are

$$x = 217°\,47'\,40'',\ 352°\,26'\,04''.$$

Check. By (3)

$$\begin{aligned} \log c &= 0.13969\,n \\ \operatorname{colog} b &= 0.03021\,n \\ \log\sin\phi &= 9.41621\,n \\ \hline \log\sin(x+\phi) &= 9.58611\,n \end{aligned}$$

EXAMPLE 2. To solve the equation

$$a\tan x + b\cot x = c.$$

Solution. Expressing the tangent and cotangent each in terms of the sine and cosine, the equation reduces to (Problem 19, Exercise 50)

$$c\sin 2x + (a - b)\cos 2x = a + b.$$

This equation is of the form

$$a\sin x + b\cos x = c,$$

which has been solved in Example 1.

EXAMPLE 3. Solve the equation

$$\sin(\alpha + x) = m\sin x. \tag{1}$$

Solution. By composition and division the proportion

$$\frac{\sin(\alpha+x)}{\sin x} = \frac{m}{1}$$

gives rise to

$$\frac{\sin(\alpha+x)+\sin x}{\sin(\alpha+x)-\sin x} = \frac{2\sin(\alpha+\frac{1}{2}x)\cos\frac{1}{2}\alpha}{2\cos(\alpha+\frac{1}{2}x)\sin\frac{1}{2}\alpha}$$

$$= \tan(\alpha+\tfrac{1}{2}x)\cot\tfrac{1}{2}\alpha = \frac{m+1}{m-1},$$

* This n indicates that the number to which the logarithm belongs is negative.

† Since ϕ is an auxiliary angle which is not retained in the end, any one of its values may be used.

or
$$\tan(\alpha + \tfrac{1}{2}x) = \frac{m+1}{m-1}\tan\tfrac{1}{2}\alpha\,. \tag{2}$$

From (2) $\alpha + \frac{1}{2}x$ and hence x can be found.

If $m = \tan\phi$

$$\frac{m+1}{m-1} = -\frac{\tan\phi + \tan\frac{1}{4}\pi}{1 - \tan\frac{1}{4}\pi\tan\phi} = -\tan(\phi + \tfrac{1}{4}\pi) = \cot(\phi - \tfrac{1}{4}\pi),$$

hence
$$\tan(\alpha + \tfrac{1}{2}x) = \cot(\phi - \tfrac{1}{4}\pi)\tan\tfrac{1}{2}\alpha, \text{ where } \tan\phi = m. \tag{3}$$

Many computers prefer (3) to (2) in solving equation (1).

EXAMPLE 4. Solve the equation

$$\tan(\alpha + x) = m\tan x.$$

Solution. Taking the proportion

$$\frac{\tan(\alpha + x)}{\tan x} = \frac{m}{1}$$

by composition and division, we obtain

$$\frac{\tan(\alpha + x) + \tan x}{\tan(\alpha + x) - \tan x} = \frac{m+1}{m-1}.$$

But by Problem 20, Exercise 51,

$$\frac{\tan(\alpha + x) + \tan x}{\tan(\alpha + x) - \tan x} = \frac{\sin(\alpha + 2x)}{\sin\alpha},$$

hence
$$\sin(\alpha + 2x) = \frac{m+1}{m-1}\sin\alpha, \tag{1}$$

or we may write the result in the form
$$\sin(\alpha + 2x) = \cot(\phi - \tfrac{1}{4}\pi)\sin\alpha, \text{ where } \tan\phi = m. \tag{2}$$

Either (1) or (2) may be used to find $\alpha + 2x$ and hence x.

EXAMPLE 5. Solve the equation

$$\tan(\alpha + x)\tan x = m.$$

Solution. Expressing the tangents in terms of sines and cosines, we find

$$\sin(\alpha + x)\sin x = m\cos(\alpha + x)\cos x,$$

which by Problem 10, Exercise 51, may be written

$$\cos\alpha - \cos(\alpha + 2x) = m[\cos(\alpha + 2x) + \cos\alpha],$$

from which

$$\cos(\alpha + 2x) = \frac{1-m}{1+m}\cos\alpha. \qquad (1)$$

By Example 3, $\dfrac{m+1}{m-1} = \cot(\phi - \frac{1}{4}\pi)$, where $\tan\phi = m$,

hence $\dfrac{m-1}{m+1} = \tan(\phi - \frac{1}{4}\pi)$,

and $\dfrac{1-m}{1+m} = -\tan(\phi - \frac{1}{4}\pi) = \tan(\frac{1}{4}\pi - \phi)$,

so that equation (1) may be written

$$\cos(\alpha + 2x) = \tan(\tfrac{1}{4}\pi - \phi)\cos\alpha, \text{ where } \tan\phi = m. \qquad (2)$$

From (1) or from (2) $\alpha + 2x$ may be found and hence x.

Exercise 55

1. If $\cos(x+\alpha) = m\cos x$, show that x is given by the formula
$\tan(\alpha + \frac{1}{2}x) = \tan(\frac{1}{4}\pi - \phi)\cot\frac{1}{2}\alpha$, where $\tan\phi = m$.

2. If $\sin(x+\alpha) = m\cos x$, show that x is given by the formula
$\tan(\frac{1}{2}\alpha + \frac{1}{4}\pi + x) = \cot(\frac{1}{4}\pi - \phi)\tan(\frac{1}{4}\pi - \frac{1}{2}\alpha)$, where $\tan\phi = m$.

3. If $\cos(x+\alpha) = m\sin x$, show that x is given by the formula
$\tan(\frac{1}{2}\alpha - \frac{1}{4}\pi + x) = \tan(\frac{1}{4}\pi - \phi)\cot(\frac{1}{4}\pi + \frac{1}{2}\alpha)$, where $\tan\phi = m$.

4. If $\tan(x+\alpha) = m\cot x$, show that
$\cos(\alpha + 2x) = \tan(\frac{1}{4}\pi - \phi)\cos\alpha$, where $\tan\phi = m$.

5. If $\cot(x+\alpha) = m\cot(x-\alpha)$, then
$\sin 2x = \cot(\phi - \frac{1}{4}\pi)\sin 2\alpha$, where $\tan\phi = m$.

6. If $\tan(\alpha + x)\cot x = m$, show that
$\sin(\alpha + 2x) = \cot(\phi - \frac{1}{4}\pi)\sin\alpha$, where $\tan\phi = m$.

7. If $\tan(\alpha + x)\tan(\alpha - x) = m$, show that
$\cos 2x = \cot(\frac{1}{4}\pi - \phi)\cos 2\alpha$.

8. Find the angles between $0°$ and $360°$ which satisfy the equation

$$4\sin x + 3\cos x = 5.$$

Ans. $x = 53° 07' 45''$.

9. Find θ from the equation

$$2.76\cos\theta - 2.32\sin\theta = 1.91.$$

Ans. $\theta = 17° 59.6', 261° 55'$.

10. Find the general solution of the equation

$$\sqrt{3}\sin x - \cos x = \sqrt{2}.$$

Ans. $x = 2n\pi + \frac{5}{12}\pi$, or $(2n+1)\pi - \frac{\pi}{12}$.

11. Find the general solution of the equation

$$(1+\sqrt{3})\tan x + (1-\sqrt{3})\cot x = 2.$$

$$x = n\pi + \tfrac{1}{4}\pi,\ n\pi - \frac{\pi}{12}.$$

12. Find r and x from the equations

$$r\sin(\alpha + x) = m,$$
$$r\sin(\beta + x) = n.$$

Suggestion. Form the sum and difference of these equations, and change each into a product by the formulas in Article 113. Dividing the first result by the second gives

$$\tan\left(\frac{\alpha+\beta}{2} + x\right) = \frac{m+n}{m-n}\tan\frac{\alpha-\beta}{2}$$

or

$$\tan\left(\frac{\alpha+\beta}{2} + x\right) = \tan(\tfrac{1}{4}\pi + \phi)\tan\frac{\alpha-\beta}{2},\ \text{where } \tan\phi = \frac{n}{m}.$$

13. If $r\cos(\alpha + x) = m$, $r\cos(\beta + x) = n$, then x is given by

$$\cot\left(\frac{\alpha+\beta}{2} + x\right) = \tan(\tfrac{1}{4}\pi + \phi)\tan\frac{\alpha-\beta}{2},\ \text{where } \tan\phi = \frac{m}{n}.$$

14. Adapt the formula

$$\tan x = \frac{m\sin\alpha}{1 + m\cos\alpha}$$

to computation by logarithms.

Suggestion. Express the tangent in terms of the sine and cosine and clear fractions, the result may be written

$$\sin x = m\sin(\alpha - x),$$

which by Example 3 may be transformed into

$$\tan(x - \tfrac{1}{2}\alpha) = \frac{m-1}{m+1}\tan\tfrac{1}{2}\alpha = \tan(\phi - \tfrac{1}{4}\pi)\tan\tfrac{1}{2}\alpha,\ \text{where } \tan\phi = m.$$

15. Show that

$$a + b = \frac{a}{\cos^2\phi},\quad \text{where}\quad \tan^2\phi = \frac{b}{a},$$

$$a - b = a\cdot\cos^2\phi,\ \text{where } \sin^2\phi = \frac{b}{a}.$$

These formulas enable us to find the logarithm of any sum or difference. Thus,

$$\log(a+b) = \log a + 2 \operatorname{colog} \cos\phi, \quad \log\tan\phi = \tfrac{1}{2}(\log b - \log a).$$
$$\log(a-b) = \log a + 2 \log \cos\phi, \quad \log\sin\phi = \tfrac{1}{2}(\log b - \log a).$$

16. Establish the following transformations:

$$\left.\begin{aligned} a \sin\alpha + b \sin\alpha &= \frac{a\cos(\alpha-\theta)}{\cos\theta} \\ a\cos\alpha - b\sin\alpha &= \frac{a\cos(\alpha+\theta)}{\cos\theta} \end{aligned}\right\}, \quad \text{where } \tan\theta = \frac{b}{a}.$$

124. Inverse Functions. The two expressions

$$y = \sin x \qquad (1)$$
$$x = \sin^{-1} y \qquad (2)$$

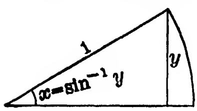


Fig. 151.

represent different views of the same relation. The first one states that y is the sine of x, the second that x is the angle or arc (measuring the angle) whose sine is y. (1) expresses y in terms of x, (2) expresses x in terms of y.

Taking the sine of each member of (2) gives us

$$\sin x = \sin(\sin^{-1} y) = y, \text{ by (1)}, \qquad (3)$$

and taking the inverse sine of each member of (1) gives us

$$\sin^{-1} y = \sin^{-1}(\sin x) = x, \text{ by (2)}. \qquad (4)$$

In precisely the same way it may be shown that

$$\cos(\cos^{-1} x) = x, \quad \cos^{-1}(\cos x) = x,$$
$$\tan(\tan^{-1} x) = x, \quad \tan^{-1}(\tan x) = x,$$
$$\text{etc.,} \qquad \text{etc.}$$

From these relations it appears that of each pair of operations, say for example that of taking the sine and that of taking the arc-sine, either undoes the other, that is, if the two operations are performed in succession, the result is that the quantity operated on is left unchanged. This explains why $\sin^{-1}x$ is called the inverse sine of x, $\tan^{-1}x$ the inverse tangent of x, etc., in fact in any pair of such functions each is the inverse of the other. Viewed as operations, the relation of the members of each pair is like that of addition to subtraction, of multiplication to division, of involution to evolution.

In general, if $y = f(x)$ represents any function of x, the inverse function is represented by $f^{-1}(x)$, and the relation between the two is always such that, considered as operations on x, each undoes the other, that is,

$$f[f^{-1}(x)] = x, \quad \text{and} \quad f^{-1}[f(x)] = x. \tag{5}$$

If $y = f(x)$ is known, $f^{-1}(x)$ is found by solving (provided we can solve) $y = f(x)$ for x, and by substituting in the result x for y. Thus, if

$$y = \frac{x^2 - 4}{3} = f(x), \tag{6}$$

we find
$$x = \sqrt{3y + 4} = f^{-1}(y),$$

and
$$\sqrt{3x + 4} = f^{-1}(x). \tag{7}$$

To verify the relations (5) we substitute for x in (6) the expression (7), and for x in (7) the expression (6); thus,

$$f[f^{-1}(x)] = \frac{(\sqrt{3x+4})^2 - 4}{3} = x,$$

$$f^{-1}[f(x)] = \sqrt{3\left(\frac{x^2 - 4}{3}\right) + 4} = x.$$

There is this important difference between the trigonometric functions and their inverses, — while each of the former has a single value, each of the latter has an indefinite number of values. Thus, if $x = \frac{\pi}{6}$, $\sin x$ has a single value, namely $\frac{1}{2}$, but if $x = \frac{1}{2}$, $\sin^{-1} x$ may have any of the values $\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{13\pi}{6}$, $\frac{17\pi}{6}$, etc. In general,

$$\sin^{-1} x = n\pi + (-1)^n \alpha,$$
$$\cos^{-1} x = 2n\pi \pm \alpha,$$
$$\tan^{-1} x = n\pi + \alpha,$$

where α is the principal value of the angle.

Any relation between trigonometric functions may be expressed by means of inverse functions. We shall illustrate the method by some examples.

EXAMPLE 1. Express the relation

$$\cos 2A = 1 - 2\sin^2 A$$

in terms of inverse functions.

Solution. Let $\sin A = m$, then $A = \sin^{-1} m$, and the given equation becomes

$$\cos(2\sin^{-1} m) = 1 - 2m^2,$$

or

$$2\sin^{-1} m = \cos^{-1}(1 - 2m^2).$$

EXAMPLE 2. Express the formula

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

in terms of inverse functions.

Solution. Let $\qquad \sin A = m, \quad \sin B = n,$

then $\qquad \cos A = \sqrt{1 - m^2}, \quad \cos B = \sqrt{1 - n^2},$

and the given formula becomes

$$\sin(\sin^{-1} m + \sin^{-1} n) = m\sqrt{1 - n^2} + n\sqrt{1 - m^2},$$

or

$$\sin^{-1} m + \sin^{-1} n = \sin^{-1}(m\sqrt{1 - n^2} + n\sqrt{1 - m^2}).$$

EXAMPLE 3. Express the formula

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

in terms of inverse functions.

Solution. Let $\qquad \tan A = m, \quad \tan B = n,$

then $\qquad A = \tan^{-1} m, \quad B = \tan^{-1} n.$

Substituting

$$\tan(\tan^{-1} m + \tan^{-1} n) = \frac{m + n}{1 - mn},$$

or

$$\tan^{-1} m + \tan^{-1} n = \tan^{-1}\frac{m + n}{1 - mn}.$$

Formulas involving inverse functions may be verified by reversing the process illustrated in the above examples.

EXAMPLE 4. Show that

$$\tan^{-1} m + \tan^{-1} n = \tan^{-1}\frac{m + n}{1 - mn}.$$

Solution. Put $\qquad \tan^{-1} m = A, \quad \tan^{-1} n = B,$

then $\qquad m = \tan A, \quad n = \tan B.$

Substituting, we find

$$A + B = \tan^{-1}\frac{\tan A + \tan B}{1 - \tan A \tan B},$$

or

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

This latter expression we know is true, hence the original expression is also true.

EXAMPLE 5. Find the value of

$$\tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{3}.$$

Solution. Put $\tan^{-1}\frac{1}{2} = A,\quad \tan^{-1}\frac{1}{3} = B,$

then $\tan A = \frac{1}{2},\quad \tan B = \frac{1}{3},$

and the given expression becomes $A + B$.

Now $$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{1}{2}+\frac{1}{3}}{1 - \frac{1}{2}\times\frac{1}{3}} = 1,$$

therefore $$A + B = \tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{3} = n\pi + \frac{\pi}{4}.$$

EXAMPLE 6. Solve the equation

$$\sin^{-1} 2x + \sin^{-1} 3x = \cos^{-1}(-\tfrac{2}{3}).$$

Solution. Put $\sin^{-1} 2x = A,\ \sin^{-1} 3x = B,$

then $\sin A = 2x,\ \sin B = 3x,$

$\cos A = \sqrt{1-4x^2},\ \cos B = \sqrt{1-9x^2},$

and the given expression becomes

$$A + B = \cos^{-1}(-\tfrac{2}{3}).$$

Take the cosine of both sides,

$$\cos(A+B) = \cos A \cos B - \sin A \sin B = -\tfrac{2}{3},$$

and expressing this in terms of x,

$$\sqrt{1-4x^2}\cdot\sqrt{1-9x^2} - 2x\cdot 3x = -\tfrac{2}{3}.$$

Solving for x,

$$x = \pm\tfrac{1}{3}.$$

EXERCISE 56

1. Find the general value of each of the following angles:

$\sin^{-1}\frac{1}{2},\ \cos^{-1}\frac{\sqrt{2}}{2},\ \tan^{-1}\sqrt{3},\ \cos^{-1} 0,\ \sec^{-1} 1,\ \tan^{-1}\infty.$

Ans. $n\pi + (-1)^n\frac{\pi}{6},\quad 2n\pi \pm \frac{\pi}{4},\quad n\pi + \frac{\pi}{3}$, etc.

Considering principal values only, verify the following:

2. $\tan^{-1}\frac{3}{2} - \tan^{-1}\frac{1}{5} = \frac{\pi}{4}.$

3. $\tan^{-1} m + \tan^{-1}\frac{1}{m} = \frac{\pi}{2}.$

4. $\cos^{-1}\frac{8}{17} + \cos^{-1}\frac{15}{17} = \frac{\pi}{2}.$

5. $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{13} = \frac{\pi}{4}.$

Express the following formulas in terms of inverse functions:

6. $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}.$

Ans. $2\tan^{-1} m = \tan^{-1}\left(\dfrac{2m}{1 - m^2}\right)$, where $m = \tan\theta$.

7. $\cos(A - B) = \cos A\cos B + \sin A\sin B.$

Ans. $\sin^{-1} m - \sin^{-1} n = \cos^{-1}(mn + \sqrt{(1 - m^2)(1 - n^2)})$, where $m = \sin A$, $n = \sin B$.

8. $\sin 2x = 2\sin x\cos x.$

Ans. $2\sin^{-1} m = \sin^{-1}(2m\sqrt{1 - m^2})$, where $m = \sin x$.

9. $\sin 3x = 3\sin x - 4\sin^3 x.$

Ans. $3\sin^{-1} m = \sin^{-1}(3m - 4m^3)$, where $m = \sin x$.

10. Show that

$$\sin^{-1} m = \cos^{-1}\sqrt{1 - m^2} = \tan^{-1}\frac{m}{\sqrt{1 - m^2}} = \sec^{-1}\frac{1}{\sqrt{1 - m^2}}$$
$$= \csc^{-1}\frac{1}{m} = \cot^{-1}\frac{\sqrt{1 - m^2}}{m}.$$

11. Show that

$$\sin^{-1}\tfrac{3}{5} + \cos^{-1}\tfrac{12}{13} = \tan^{-1}\tfrac{56}{33}.$$

12. Show that

$$2\tan^{-1}\tfrac{1}{2} + 3\tan^{-1}\tfrac{1}{3} = \tan^{-1}(-3).$$

Find x in the following equations:

13. $\sin^{-1} 2x + \sin^{-1} x = \frac{\pi}{3}.$ *Ans.* $x = \pm\dfrac{\sqrt{21}}{14}.$

14. $\tan^{-1}(1 + x) + \tan^{-1}(1 - x) = \tan^{-1}\frac{2}{25}.$ *Ans.* $x = \pm 5.$

15. $\sin^{-1} x + 2\cos^{-1} x = \tan^{-1}\sqrt{3}.$ *Ans.* $x = \pm\dfrac{\sqrt{3}}{2}.$

16. Show that if $f(x) = \dfrac{x+1}{x}$, then $f^{-1}(x) = \dfrac{1}{x-1}$.

17. Prove that the inverse of $x^2 + 4$ is $\sqrt{x-4}$.

18. What is the inverse of $\log_{10} x$? Of $1 - x$?

Ans. 10^x, $1 - x$.

19. Prove that $\dfrac{x+1}{x-1}$ is its own inverse.

Find the inverse functions of each of the following, and verify the results:

20. $f(x) = \dfrac{x^3}{3}$.

21. $f(y) = \dfrac{y^2 + 2y}{5}$.

22. $f(\theta) = \dfrac{\theta^2 + 4}{\theta^2 - 4}$. *Ans.* $f^{-1}(\theta) = 2\sqrt{\dfrac{\theta+1}{\theta-1}}$.

125. Review. 1. (*a*) Define the sine, cosine and tangent of any angle. (*b*) Give the signs of the principal functions in each of the four quadrants. (*c*) Give the formula which expresses the periodicity of the sine. (*d*) Prove that $\tan(\theta + n\pi) = \tan\theta$. (*e*) Which other function has the same period as the tangent?

2. (*a*) Follow the changes in the sine of an angle as the angle increases from 0 to 2π. (*b*) Do the same for the cosine. (*c*) Do the same for the tangent. (*d*) Follow the changes in the reciprocal of the tangent.

3. (*a*) Draw the lines which represent the various functions of an arc of a circle (or of its angle at the center of the circle) when the radius of the circle is taken as the unit of measure. (*b*) Draw these lines for an angle in the third quadrant. (*c*) Explain the derivation of the words secant, tangent, sine and cosine.

4. (*a*) Prove geometrically that $\sin(R + \theta) = \cos\theta$, $\tan(3R - \theta) = \cot\theta$, θ being an angle in the first quadrant. (*b*) Prove that $\sin(-\theta) = -\sin\theta$, $\cos(2R + \theta) = -\cos\theta$ for every value of θ. (*c*) Give from memory the principal functions of 30°, 150°, 210°, 330°. (*d*) Give from memory the principal functions of 45°, 135°, 225°, 315°.

5. (*a*) Show that

$$\frac{\sin(-A) + \cos(-A)}{\tan(-A) - \cot(-A)} = \frac{\sin(90^\circ + A) + \cos(270^\circ - A)}{\cot(180^\circ + A) + \tan(360^\circ - A)}.$$

(*b*) Simplify $\dfrac{\sin(R+x)\cos(R-x)}{\cos(2R+x)}+\dfrac{\sin(R-x)\cos(R+x)}{\sin(2R+x)}$.

(*c*) Find from the tables sin 234°, cos 342°, tan 134° 54′, sin 967° 45′, $\tan\left(n\pi+\dfrac{\pi}{10}\right)$.

6. (*a*) Prove the addition theorem for the sine, for the cosine and for the tangent. (*b*) Give the formulas for $\sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta$. (*c*) Show that $\sin(x+y)\sin(x-y)=(\sin x+\sin y)(\sin x-\sin y)$. (*d*) Given $\cos x=\frac{4}{5}$, find $\sin\dfrac{x}{2}$, $\cos\dfrac{x}{2}$ and $\tan\dfrac{x}{2}$ without the use of tables.

7. Given the law of sines, prove the law of tangents.

8. Express each of the following as a product:

$$\sin 7^\circ+\sin 15^\circ,\ \cos\frac{x}{2}-\cos\frac{3x}{2},\ \sin A+\sin 3A+\sin 4A.$$

9. By multiplying each side of the expression

$$S=1+\cos x+\cos 2x+\cos 3x+\ldots+\cos nx, \text{ by } 2\sin\frac{x}{2}$$

and expressing each product on the right as a difference of two sines show that

$$S=\frac{\sin\dfrac{(2n+1)x}{2}+\sin\dfrac{x}{2}}{2\sin\left(\dfrac{x}{2}\right)}=\frac{\sin\dfrac{(n+1)x}{2}\cos\dfrac{(nx)}{2}}{\sin\left(\dfrac{x}{2}\right)}.$$

10. (*a*) What is meant by the principal value of an angle? (*b*) Give the principal values of $\sin^{-1}\frac{1}{2}$, $\sin^{-1}-\frac{1}{2}$, $\cos^{-1}\frac{1}{2}$, $\tan^{-1}-1$. (*c*) Give the general values of the angles under (*b*).

11. (*a*) What is meant by a trigonometric equation? (*b*) Solve the following equations, $\sin x=-\cos x$, $2\sin^2 x+3\cos^2 x=5$, $a\sin x=b\tan x$.

12. Solve the equations:

(*a*) $\sin(x+c)-\cos x\sin c=\cos c$.

(*b*) $\sin(a-x)=\cos(a+x)$.

(*c*) $\sin(x+y)=\cos(x-y)=\dfrac{\sqrt{2}}{2}$.

13. Solve the equations:

(*a*) $\sin 2y + \sin 3y = 3 \sin y$. (*b*) $1 + \cos x = \cos \frac{x}{2}$.

(*c*) $\sin x \cos x = \frac{1}{4}$. (*d*) $\sin x + \cos x = \sqrt{2}$.

14. (*a*) Show how to adapt the equation $a \sin x + b \cos x = c$ to solution by means of logarithms. (*b*) Solve the simultaneous equations: $x \cos \alpha + y \sin \alpha = a$, $x \sin \alpha + y \cos \alpha = b$.

15. (*a*) Define the inverse trigonometric functions.

(*b*) Complete the following equalities, $\sin^{-1} x = \cos^{-1}(\quad) = \tan^{-1}(\quad) = \sec^{-1}(\quad) = \csc^{-1}(\quad) = \cot^{-1}(\quad)$.

(*c*) Find $\tan(\sin^{-1} \frac{4}{5})$, $\sin(\tan^{-1} x)$.

(*d*) Show that $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = 45°$.

16. (*a*) Prove that $\sin(2 \sin^{-1} x) = 2x\sqrt{1 - x^2}$.

(*b*) $\sin^{-1} \frac{4}{5} + \sin^{-1} \frac{8}{17} + \sin^{-1} \frac{13}{85} = \frac{\pi}{2}$.

17. (*a*) If $f(x) = \frac{x^2 + 1}{x^2 - 1}$, find $f^{-1}(x)$.

(*b*) Solve the equation,

$$\cos^{-1} x + \cos^{-1}(1 - x) = \cos^{-1}(-x).$$

CHAPTER XIII

TRIGONOMETRIC CURVES

126. Functions Represented by Curves. The student is probably already familiar with the fact that for every function of $x, f(x)$, a curve or graph may be constructed which is said to represent that function. This curve is merely the totality of all the points whose coördinates x, y, satisfy the relation $y = f(x)$. The actual construction of the curve representing a given function $y = f(x)$ consists in plotting a limited number of points, using for abscissas properly chosen values of x, and for ordinates the corresponding values of y determined by the relation $y = f(x)$. The smooth curve connecting the points thus plotted is said to be the curve or graph representing the function $y = f(x)$.

127. The Straight Line, $y = mx + c$.

Suppose the given function is

$$y = 2x + 1.$$

We give x certain values and compute the corresponding values of y. Thus, if

$$\begin{array}{llllllllllll} x = & 5, & 4, & 3, & 2, & 1, & 0, & -1, & -2, & -3, & -4, & -5, \text{ etc.,} \\ y = & 11, & 9, & 7, & 5, & 3, & 1, & -1, & -3, & -5, & -7, & -9, \text{ etc.} \end{array}$$

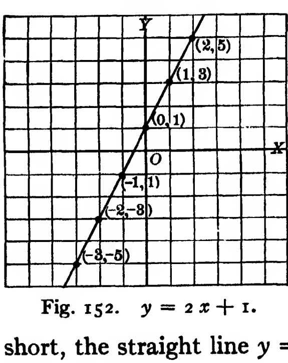


Fig. 152. $y = 2x + 1$.

We now locate the point whose abscissa is 5 and whose ordinate is 11, another point whose abscissa is 4 and whose ordinate is 9, and similarly each of the points (3, 7), (2, 5), (1, 3), (0, 1), (− 1, − 1), etc.

We then connect the points thus located by a smooth curve, in this case a straight line, Fig. 152. This line is said to be the curve or graph representing the function $y = 2x + 1$, or, in short, the straight line $y = 2x + 1$.

In the example just given, x may have one value as well as another, say 1,000,000 as well as 2 or 3, consequently the line representing the equation $y = 2x + 1$ will be indefinite in length and we must content ourselves with drawing only a portion of it. What portion this is to be depends on the purpose in view, but unless there is a special reason to the contrary, it is customary to construct the portion nearest the origin.

In a similar manner every equation of the form

$$y = mx + c,$$

where m and c are known numbers, is represented by some straight line, m and c determining the direction and position of the line with respect to the coördinate axes.

128. The Circle, $x^2 + y^2 = a^2$.

Suppose the given equation is

$$x^2 + y^2 = 25,$$

then

$$y = \pm\sqrt{25 - x^2},$$

and we have for corresponding values of x and y,

$x = 5,$	$4,$	$3,$	$2,$	$1,$	$0,$
$y = 0,$	$\pm 3,$	$\pm 4,$	$\pm 4.58,$	$\pm 4.89,$	$\pm 5,$
$x = -1,$	$-2,$	$-3,$	$-4,$	$-5,$	etc.,
$y = \pm 4.89,$	$\pm 4.58,$	$\pm 4,$	$\pm 3,$	$0,$	etc.

If, as before, we construct the separate points

$(5, 0)$, $(4, 3)$, $(4, -3)$, $(3, 4)$, $(3, -4)$, $(2, 4.58)$, $(2, -4.58)$, etc., and connect the points thus obtained by a smooth curve, we obtain the circle, Fig. 153, which is said to be the curve or graph representing the equation $x^2 + y^2 = 25$, or, in short, the circle $x^2 + y^2 = 25$. In this case x cannot be numerically greater than 5, for then y would be imaginary.

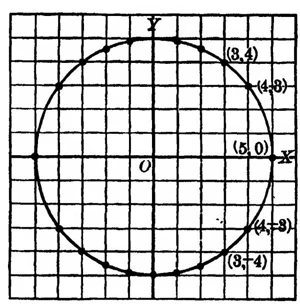


Fig. 153. $x^2 + y^2 = 25$.

In like manner every equation of the form

$$x^2 + y^2 = a^2,$$

where a is some known number, is represented by a circle whose center is at the origin and whose radius is a.

129. The Hyperbola, $x^2 - y^2 = a^2$.

As another example, let us construct the curve whose equation is

$$x^2 - y^2 = 25.$$

Solving for y,

$$y = \pm\sqrt{x^2 - 25}.$$

Corresponding values of x and y are

$x = 5,$	$6,$	$7,$	$8,$	$9,$	10
$y = 0,$	$\pm 3.32,$	$\pm 4.90,$	$\pm 6.25,$	$\pm 7.48,$	$\pm 8.66,$
$x = -5,$	$-6,$	$-7,$	$-8,$	$-9,$	etc.
$y = +0,$	$\pm 3.32,$	$\pm 4.90,$	$\pm 6.25,$	$\pm 7.48,$	etc.,

In this case x cannot be numerically less than 5, for otherwise y is imaginary.

If we construct the separate points (5, 0), (6, 3.32), (6, − 3.32), etc., and draw a smooth curve connecting them, we obtain the two curves PQ, $P'Q'$, Fig. 154. These curves constitute the two branches of a single curve, known as the hyperbola, more specifically as the *equilateral hyperbola.*

It is easy to see from the equation that the larger x is, the more nearly will x and y be equal, that is, the branches of the equilateral hyperbola approach the straight lines PP' and QQ' drawn through the origin and making angles of 45° with the two directions of the x-axis respectively.

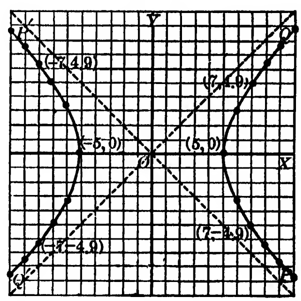


Fig. 154. $x^2 - y^2 = 25$.

In like manner it will be found that every equation of the form

$$x^2 - y^2 = a^2,$$

where a is some known number, is represented by an equilateral hyperbola. a determines the distance from the origin at which the hyperbola crosses the x-axis. This is known as the *semimajor axis* of the hyperbola.

130. The Sine Curve, $y = \sin x$.

Let us now construct the curve representing $y = \sin x$.

Since x may have any value either positive or negative, the curve representing the sine function will extend indefinitely in both direc-

tions (right and left). Let us first construct that portion of the curve which corresponds to values of x between 0 and $\frac{1}{2}\pi$, that is, to angles in the first quadrant. Referring to the table of natural sines, we find the following corresponding values of x and y:

$x =$	In degrees,	$0°$,	$10°$,	$20°$,	$30°$,	$40°$,	$50°$,	$60°$,	$70°$,	$80°$,	$90°$.
	In radians,	0,	$\frac{\pi}{18}$,	$\frac{\pi}{9}$,	$\frac{\pi}{6}$,	$\frac{2\pi}{9}$,	$\frac{5\pi}{18}$,	$\frac{\pi}{3}$,	$\frac{7\pi}{18}$,	$\frac{4\pi}{9}$,	$\frac{\pi}{2}$.
$y =$		0,	0.17,	0.34,	0.5,	0.64,	0.77,	0.87,	0.94,	0.98,	1.

In order to avoid awkward fractions, we will use $\frac{1}{3}\pi$ as the unit along the x-axis.*

With $\frac{\pi}{3}$ for a unit, $\frac{\pi}{18} = \frac{1}{6}$ unit, $\frac{\pi}{9} = \frac{1}{3}$ unit, etc., and we now readily locate the points

$$O = (0, 0), P_1 = \left(\frac{\pi}{18}, 0.17\right), P_2 = \left(\frac{\pi}{9}, 0.34\right), \ldots, R_1 = \left(\frac{\pi}{2}, 1\right).$$

Connecting these points by a smooth curve we obtain the curve $OP_1P_2P_3 \ldots R_1$ (Fig. 155), which is the sine curve for the first quadrant.

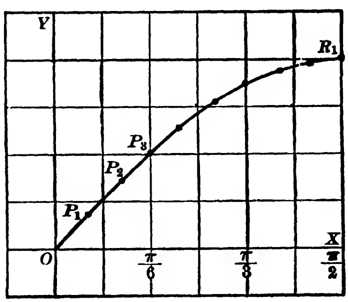


Fig. 155. $y = \sin x$.

We may now easily continue the sine curve through as many additional quadrants as we choose.

Second quadrant. While x varies from $\frac{1}{2}\pi$ to π, $\sin x$ varies from 1 to 0. Moreover, since sines of supplementary angles are equal, ordinates equally distant from R_1 will be equal, that is, the curve will be symmetrical with respect to the ordinate at R_1. Hence, con-

* This will distort the curve slightly, since $\frac{\pi}{3} = \frac{3.14159+}{3} = 1.0471+$.

tinuing from R_1, the curve will approach the x-axis, meeting it at R_2 (Fig. 156), whose distance from the origin is π.

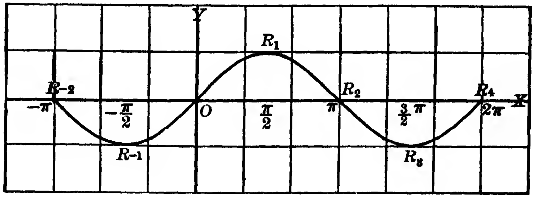


Fig. 156. $y = \sin x$.

Third and fourth quadrants. While x varies from π to 2π, $\sin x$ is negative, the numerical value being the same as when x varies from 0 to π. Hence, continuing from R_2 the curve will descend below the x-axis, reaching the lowest point at R_3, where $x = \frac{3}{2}\pi$, and meeting the x-axis again at R_4, where $x = 2\pi$. The form of the portion of the curve below the x-axis will be like that above the x-axis when revolved about this axis through an angle of 180°.

When x is increased or diminished by 2π, $\sin x$ has the same value as before, hence extending from R_4 to the right or from O to the left the curve repeats itself indefinitely, that is, the complete sine curve consists of an infinite number of waves or undulations of which $OR_1R_2R_3R_4$ is one.

OR_4, the distance between two consecutive points at which the curve crosses the x-axis in the same direction, is called the *wave length* of the curve. The greatest height of the curve, represented by the ordinate at R_1, is called the *amplitude* of the curve.

A curve like the sine curve, which repeats itself at definite intervals, is called a *periodic curve;* the interval at which the repetition takes place is called the *period.* Likewise the function which such a curve represents is called a *periodic function.*

The sine function is a periodic function whose period is 2π.

131. The Tangent Curve, $y = \tan x$.

To construct the tangent curve for the first quadrant, we compute by means of a table of natural functions the corresponding values of x and y, as follows:

$$x = \begin{cases} \text{In degrees, } 0°, 10°, 20°, 30°, 40°, 50°, 60°, 70°, 80°, 90°. \\ \text{In radians, } 0, \frac{\pi}{18}, \frac{\pi}{9}, \frac{\pi}{6}, \frac{2\pi}{9}, \frac{5\pi}{18}, \frac{\pi}{3}, \frac{7\pi}{18}, \frac{4\pi}{9}, \frac{\pi}{2}. \end{cases}$$

$$y = 0, 0.18, 0.36, 0.58, 0.84, 1.19, 1.73, 2.75, 5.67, \infty.$$

Plotting the points obtained by using the x's for abscissas and the corresponding values of y for ordinates, and connecting these points by a smooth curve, we obtain the curve OR, Fig. 157, which represents the equation $y = \tan x$ for values of x in the first quadrant. Since $\tan \frac{1}{2}\pi = \infty$, the curve will not intersect the perpendicular at $\frac{1}{2}\pi$, but will approach it indefinitely.

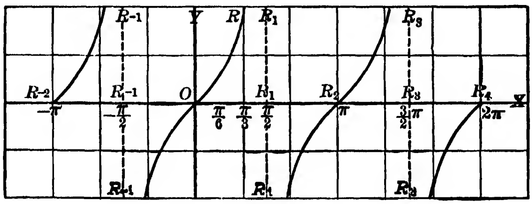


Fig. 157. $y = \tan x$.

Second quadrant. While x varies from $\frac{1}{2}\pi$ to π, the tangent varies from $-\infty$ to 0, hence between R_1 and R_2 the tangent curve will be below the x-axis, the numerical values of the ordinates being equal to those between O and R_1 taken in the reverse order.

Tan $(\pi + x) = \tan x$, hence beginning with R_2 the curve repeats itself. The tangent curve, therefore, consists of an infinite number of disconnected branches.

The tangent curve is a periodic curve, the tangent function is a periodic function, the period in each case is π.

A line like R_1R_1 or $R_{-1}R_{-1}$, Fig. 157, to which the curve approaches indefinitely near without ever reaching it, is called an *asymptote* to the curve. The lines PP' and QQ', Fig. 154, are asymptotes to the hyperbola. An asymptote is a tangent to the curve at infinity.

When the student has become familiar with the forms of the sine and tangent curves, he can readily sketch them from a very few points whose coördinates are known from memory. Thus, for the values

$$x = 0, \quad \frac{\pi}{6}, \quad \frac{\pi}{4}, \quad \frac{\pi}{3}, \quad \frac{\pi}{2}$$

the corresponding values of the functions are known without consulting a table, namely

$$\sin x = 0, \quad 0.5, \quad \frac{\sqrt{2}}{2} = 0.71, \quad \frac{\sqrt{3}}{2} = 0.87, \quad 1,$$

$$\tan x = 0, \quad \frac{\sqrt{3}}{3} = 0.58, \quad 1, \quad \sqrt{3} = 1.73, \quad \infty.$$

The same remark applies to the sketching of each of the remaining trigonometric functions.

Exercise 57

1. Construct the cosine curve.

2. Construct the cosecant curve.

(Suggestion. This curve is most readily sketched from the sine curve by remembering that $y = \csc x = \frac{1}{\sin x}$.)

3. Construct the secant curve. $\left(y = \sec x = \frac{1}{\cos x}\right)$.

4. Construct the cotangent curve. $\left(y = \cot x = \frac{1}{\tan x}\right)$.

5. Construct the curves whose equations are $y = \sin^{-1} x$, $y = \tan^{-1} x$.

(Suggestion. If $y = \sin^{-1} x$, then $x = \sin y$, etc.)

6. $y = \cos x = \sin\left(\frac{\pi}{2} + x\right)$. From this relation it follows that for every ordinate on the sine curve there is an equal ordinate on the cosine curve whose abscissa is the abscissa of the former diminished by $\frac{1}{2}\pi$, that is, for every point on the sine curve there is a point on the cosine curve, the latter being $\frac{1}{2}\pi$ to the left of the former. Thus we see that the cosine curve is merely the sine curve shifted a distance $\frac{1}{2}\pi$ to the left. By a similar reasoning show that the secant curve is the cosecant curve shifted a distance $\frac{1}{2}\pi$ to the left.

7. $y = \cot x = -\tan\left(\frac{\pi}{2} + x\right)$. From this relation show that the cotangent curve may be obtained by shifting the tangent curve a distance of $\frac{1}{2}\pi$ to the left, and revolving it about the x-axis through an angle of $180°$.

8. $y = \cos x = \sin\left(\frac{\pi}{2} - x\right) = -\sin\left(x - \frac{\pi}{2}\right)$. From this relation show that the cosine curve may be obtained by shifting the sine

curve a distance of $\frac{1}{2}\pi$ to the right, and revolving it about the x-axis through an angle of 180°.

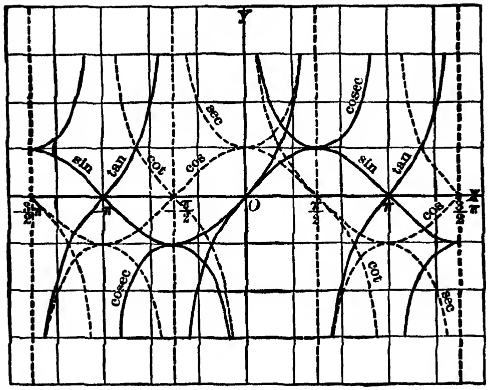


Fig. 158. The Six Trigonometric Curves.

132. The Sinusoidal or Simple Harmonic Curves,

$$y = a \sin (kx + \epsilon).$$

(*a*) $y = a \sin x$, a being constant. Each ordinate of the curve representing this equation is a times the corresponding ordinate of the curve $y = \sin x$. The required curve is the curve obtained by lengthening or shortening (according as a is greater or less than

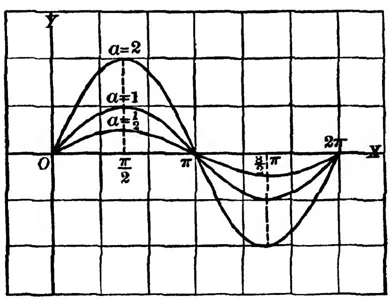


Fig. 159. $y = a \sin x$.

unity) the sine curve in the direction of the y-axis, leaving the wave length unchanged. The curve is a sinelike curve (sinusoid) whose amplitude is a. Fig. 159 shows three sinusoids of equal wave lengths and amplitudes $\frac{1}{2}$, 1, 2 respectively.

(*b*) $y = a \sin kx$, a and k being constants.

The curve representing this equation has the same amplitude as the curve $y = a \sin x$, but the wave length differs, for it crosses the x-axis when

$$kx = 0, \quad \pi, \quad 2\pi, \quad \text{etc.},$$

that is, when

$$x = 0, \quad \frac{\pi}{k}, \quad \frac{2\pi}{k}, \quad \text{etc.}$$

The distance between two consecutive crossings of the x-axis in the same direction is $2\pi/k$, that is, the required curve is a sinusoid whose wave length is $2\pi/k$. If we denote this wave length by λ, we have

$$\lambda = \frac{2\pi}{k}, \text{ from which } k = \frac{2\pi}{\lambda},$$

and the equation $\quad y = a \sin kx \quad$ may be written

$$y = a \sin \frac{2\pi x}{\lambda}, \quad \lambda \text{ being the wave length.}$$

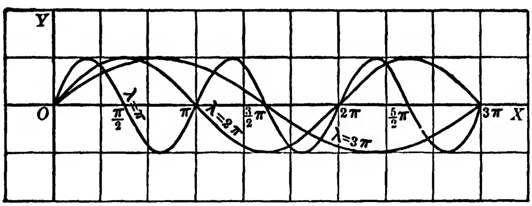


Fig. 160. $y = a \sin (2\pi x/\lambda)$.

Fig. 160 shows three sinusoids of equal amplitudes and wave lengths π, 2π, 3π respectively.

(*c*) $y = a \sin (kx + \epsilon)$, a, k and ϵ being constants.

The curve representing this equation crosses the x-axis where

$$kx + \epsilon = \quad 0, \quad \pi, \quad 2\pi, \quad \text{etc.},$$

that is, where

$$x = -\frac{\epsilon}{k}, \quad \frac{\pi - \epsilon}{k}, \quad \frac{2\pi - \epsilon}{k}, \text{ etc.}$$

The ordinates of the highest and lowest points are a and $-a$ respectively, and the distance between two points where the curve crosses the x-axis in the same direction is $2\pi/k$. When $x = 0$, $y = a \sin \epsilon$. The curve is a sinusoid, amplitude a, wave length $2\pi/k$,

but instead of passing through the origin, it crosses the y-axis at a distance $a \sin \epsilon$ above the origin. Fig. 161 shows three sinusoids of

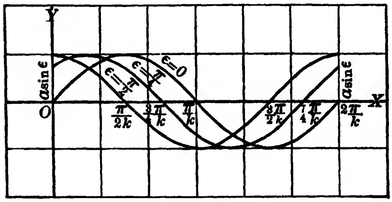


Fig. 161.

equal amplitudes and wave lengths and $\epsilon = 0$, $\frac{1}{4}\pi$ and $\frac{1}{2}\pi$ respectively. The third of these curves has for its equation

$$y = a \sin\left(kx + \frac{\pi}{2}\right) = a \cos kx,$$

which shows that every cosine curve is a sinusoid.

Sinusoids are extensively used in physics to represent the motion of vibrating strings, tuning forks, and other vibrating bodies emitting musical sounds. For this reason they are often referred to as *harmonic*, or, more strictly, as *simple harmonic curves*.

133. Angles as Functions of Time. In many physical problems which give rise to equations of the form $y = a \sin(kx + \epsilon)$, the independent variable represents not an angle, but the time of an action or motion.

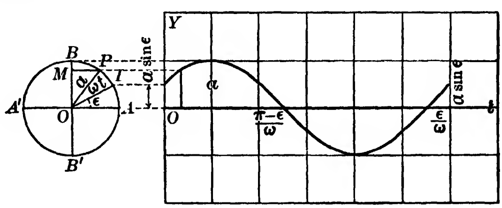


Fig. 162.

Let I, Fig. 162, be the initial position of a point moving in the circumference of a circle (radius $OP = a$) with a constant velocity. Let P be the position occupied by the moving point, t seconds after leaving I, and let ω represent the angular velocity of P, that is, the angle described by OP in one second of time. Then angle $IOP = \omega t$,

and angle $AOP = \omega t + \epsilon$, where ϵ is the angle which OI makes with some fixed diameter as AOA'. Furthermore, let M represent the projection of P on BB', the diameter perpendicular to AA', then $OM = a \sin (\omega t + \epsilon)$. The curve

$$y = a \sin (\omega t + \epsilon) \tag{1}$$

obtained by using equal distances on the axis of abscissas to represent equal intervals of time, and using for ordinates the distances OM, which correspond to various values of t, enables us to see at a glance the position of M at any given time t.

If T represents the time required by P to complete one revolution we have

$$\omega T = 2\pi, \text{ from which } \omega = \frac{2\pi}{T},$$

so that (1) may also be written

$$y = a \sin\left(\frac{2\pi t}{T} + \epsilon\right), \tag{2}$$

where T is the *periodic time*, or *period of oscillation* of P and M.

Since T represents the time required to complete one revolution, its reciprocal $1/T$ will represent the number of revolutions completed in a unit of time. This value $1/T$ is known as the *frequency of the oscillation*. If the frequency is denoted by ν, equation (2) assumes the form

$$y = a \sin (2\nu\pi t + \epsilon). \tag{3}$$

Motion like that of the point M in Fig. 162, that is, any motion which can be expressed by an equation of the form $y = a \sin (bt + c)$ is called *simple harmonic*. The motion of vibrating tuning forks, of water waves, of an oscillating pendulum, of a galvanometer needle, of alternating currents, of sound and light and magnetic electric waves, are familiar examples of simple harmonic motions.

Exercise 58

Plot the following curves:

1. $y = \sin 2x$.
2. $y = \sin \frac{x}{2}$.
3. $y = 3 \sin \frac{x}{3}$.
4. $y = \frac{1}{3} \sin 3x$.
5. $y = \sin\left(x + \frac{\pi}{4}\right)$.
6. $y = 2 \sin\left(\frac{3\pi}{2}x - \frac{\pi}{6}\right)$.

7. Construct sinusoids having the following amplitudes and wave lengths:

(*a*) Amplitude = 1.5, wave length = 3.5.

(*b*) Amplitude = 0.25, wave length = $\frac{\pi}{2}$.

(*c*) Amplitude = 1, wave length = 8π.

Write the equation for each of these curves.

Ans. (*a*) $y = 1.5 \sin \frac{4\pi x}{7}$, (*b*) $4y = \sin 4x$, (*c*) $y = \sin \frac{x}{4}$.

8. Show that the equation

$$y = a \cos (bt + c)$$

may be used to represent harmonic motion as well as the equation

$$y = a \sin (bt + c).$$

9. Plot the curves (*a*) $y = \sin^2 x$, (*b*) $y = \tan^2 x$.

10. Plot the curves (*a*) $y^2 = \sin x$, (*b*) $y^2 = \tan x$.

11. A point moves in the circumference of a circle whose radius is 8.5, from an initial position whose angular distance from the right-hand extremity of a horizontal diameter is 15°, with a uniform velocity such as to complete a single revolution in 54 seconds. Write the equation between y and t, where y represents the vertical distance of the point from the horizontal diameter at any given time t.

12. A piece of paper is wrapped around a wooden cylinder and then an oblique section is made by sawing the cylinder in two. If the paper is now unrolled and laid flat, its edge will form a sinusoidal curve. Prove it.

Ans. The equation of the curve is $y = a \sin \frac{x}{r}$, where $PR = y$, $OR = x$, $AT = a$, and $OC = r =$ the radius of the cylinder.

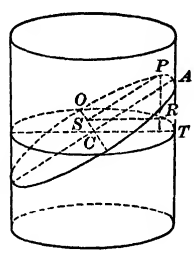


Fig. 163.

134. Composition of Sinusoidal Curves. Curves representing an equation of the form

$$y = a_1 \sin (b_1 x + c_1) + a_2 \sin (b_2 x + c_2) + \text{ etc.},$$

may be readily constructed from the component curves

$$y_1 = a_1 \sin (b_1 x + c_1), \ \ y_2 = a_2 \sin (b_2 x + c_2), \ \text{etc.},$$

for since

$$y = y_1 + y_2$$

the ordinate of any point on the required curve is found by adding the corresponding ordinates of the component curves.

EXAMPLE 1. Plot the curve $y = \sin x + \cos x$.

Solution. Plot separately the two curves

$$y_1 = \sin x, \text{ [curve (1), Fig. 164]},$$
$$y_2 = \cos x, \text{ [curve (2), Fig. 164]}.$$

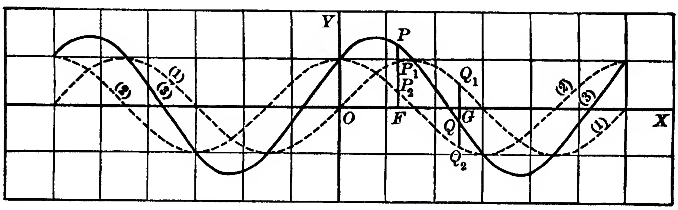


Fig. 164.

The required curve $y = y_1 + y_2$ [curve (3), Fig. 164] is then obtained by adding the two ordinates corresponding to any given value of x. Thus, if $x = OF$, $y = FP = FP_1 + FP_2$. It is important to remember that for points below the x-axis the ordinates are negative; thus, if $x = OG$, $y = GQ = GQ_1 + GQ_2$. Now GQ_1 is positive but GQ_2 is negative. GQ_2 being the longer of the two, their algebraic sum, that is, GQ, will be negative.

It should be noticed that certain points, as, for instance, those for which $y = \pm 1$, ± 2, or 0, may be located at sight. After the shape of the curve is known, these points suffice to sketch the required curve.

Each of the sine curves $y_1 = \sin x$ and $y_2 = \cos x$ has the period 2π and the resultant curve $y = \sin x + \cos x$ is another sine curve with the period 2π. This may be shown analytically as follows:

$$y = \sin x + \cos x = \sin x + \sin\left(\frac{\pi}{2} - x\right),$$

$$= 2 \sin\frac{\pi}{4} \cos\left(\frac{\pi}{4} - x\right), \text{ by formulas in Article 113},$$

$$= \sqrt{2} \cos\left(\frac{\pi}{4} - x\right) = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right),$$

that is, the curve $y = \sin x + \cos x$ is a sinusoid having an amplitude $\sqrt{2}$, period 2π, and crossing the x-axis at the point $x = -\frac{1}{4}\pi$.

The method employed in constructing the curve in the preceding example applies equally well to the compounding of any number of curves. The ordinates of the resultant curve are always the algebraic sums of the ordinates of the component curves.

EXAMPLE 2. Plot the curve $y = \sin x + \sin 2x$.

Solution. Construct separately the two curves

$$y_1 = \sin x, \quad [\text{curve } (1)],$$
$$y_2 = \sin 2x, \quad [\text{curve } (2)],$$

then

$$y = y_1 + y_2 = \sin x + \sin 2x$$

yields the curve (3), Fig. 165. This curve has the period 2π.

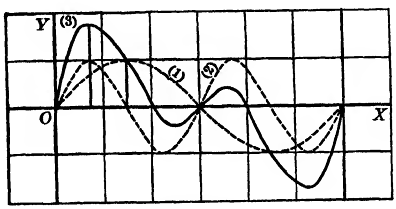


Fig. 165.

EXAMPLE 3. Plot the curve $y = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x$.

Solution.

$y_1 = \sin x$ gives curve (1).

$y_2 = \frac{1}{3}\sin 3x$ gives curve (2),

$y_3 = \frac{1}{5}\sin 5x$ gives curve (3),

$y = y_1 + y_2 + y_3$ gives curve (4).

This curve also has the period 2π.

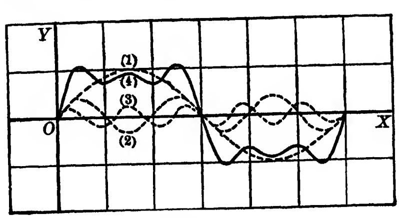


Fig. 166.

135. Theorem. *The resultant of two simple harmonic curves having equal wave lengths is another simple harmonic curve having the same wave length.*

Proof. Let the given curves be

$$y_1 = a_1 \sin\left(\frac{2\pi}{\lambda} x + c_1\right), \quad y_2 = a_2 \sin\left(\frac{2\pi}{\lambda} x + c_2\right),$$

λ being the common wave length. (Art. 132.)
Now

$$y_1 = a_1 \sin\left(\frac{2\pi}{\lambda} x\right) \cos c_1 + a_1 \cos\left(\frac{2\pi}{\lambda} x\right) \sin c_1,$$

$$y_2 = a_2 \sin\left(\frac{2\pi}{\lambda} x\right) \cos c_2 + a_2 \cos\left(\frac{2\pi}{\lambda} x\right) \sin c_2,$$

therefore

$$\begin{aligned} y &= y_1 + y_2 \\ &= (a_1 \cos c_1 + a_2 \cos c_2) \sin\left(\frac{2\pi}{\lambda} x\right) + (a_1 \sin c_1 + a_2 \sin c_2) \cos\left(\frac{2\pi}{\lambda} x\right) \\ &= a \cos c \sin\left(\frac{2\pi}{\lambda} x\right) + a \sin c \cos\left(\frac{2\pi}{\lambda} x\right) \\ &= a \sin\left(\frac{2\pi}{\lambda} x + c\right), \end{aligned} \tag{1}$$

where

$$\begin{aligned} a \cos c &= a_1 \cos c_1 + a_2 \cos c_2, \\ a \sin c &= a_1 \sin c_1 + a_2 \sin c_2. \end{aligned} \tag{2}$$

Dividing the second of the equations (2) by the first, gives

$$\tan c = \frac{a_1 \sin c_1 + a_2 \sin c_2}{a_1 \cos c_1 + a_2 \cos c_2}, \tag{3}$$

and taking the sum of their squares

$$\begin{aligned} a^2 (\cos^2 c + \sin^2 c) = a_1^2 (\cos^2 c_1 + \sin^2 c_1) &+ 2\, a_1 a_2 (\cos c_1 \cos c_2 \\ + \sin c_1 \sin c_2) &+ a_2^2 (\cos^2 c_2 + \sin^2 c_2), \end{aligned}$$

or

$$a^2 = a_1^2 + a_2^2 + 2\, a_1 a_2 \cos (c_1 - c_2). \tag{4}$$

Equation (1) shows that the resultant curve is a simple harmonic curve whose wave length is λ. The amplitude a of the resultant curve is given by (4), and the constant c by (3).

136. Fourier's Theorem. Before leaving the subject of harmonic curves, we will state in simple language and without proof a most famous theorem, which in the language of Thomson and Tait * "is not only one of the most beautiful results of modern analysis, but may be said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics. To mention only sonorous vibrations, the propagation of electric signals along telegraph wires, and the conduction of heat by the earth's crust, as subjects in their generality intractable without it, is to give but a feeble idea of its importance."

Any arbitrary periodic curve can be considered the resultant of a sum of simple harmonic curves, and can therefore be expressed by an equation of the form

$$y = a_1 \sin (b_1 x + c_1) + a_2 \sin (b_2 x + c_2) + \text{etc.}$$

The importance of the theorem lies in this, that every periodic phenomenon whose changes can be measured can be represented by a periodic curve. Fourier's theorem shows how every such phenomenon can be resolved into a series of simple harmonic motions.

Exercise 59

1. Plot the curve $y = \sin x - \cos x$.

Show analytically that this curve is a sine curve, whose amplitude is $\sqrt{2}$, and which crosses the x-axis at the distance $\frac{1}{4}\pi$ to the right of the origin.

2. Construct the resultant curve of which

$$y_1 = \cos\left(x + \frac{\pi}{3}\right), \quad \text{and} \quad y_2 = \cos\left(x - \frac{\pi}{3}\right),$$

are the components, and show that the equation of the resultant curve may be written $y = \cos x$.

Construct the following curves:

3. $y = \sin x + \frac{1}{3} \sin 3x$.
4. $y = \sin 2x + \sin 3x$.
5. $y = \sin x + \frac{1}{2} \sin 4x$.
6. $y = 2 \sin x - \sin 2x$.
7. $y = \cos 2x + 4 \cos x$.
8. $y = \sin x - \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x$.
9. $x = \tan^{-1}(1 + \sqrt{y}) + \tan^{-1}(1 - \sqrt{y})$.

* Elements of Natural Philosophy, Second Edition, Chapter I.

10. Plot the curve resulting from compounding the two curves

$$y_1 = 2 \sin\left(x + \frac{\pi}{6}\right), \qquad y_2 = 3 \sin\left(x - \frac{\pi}{6}\right),$$

and show that the equation of the resultant curve is

$$y = a \sin (x + c),$$

where $\quad a = \sqrt{19}, \;\; c = \tan^{-1}\left(-\frac{\sqrt{3}}{15}\right).$

11. Show that

$$a \sin \phi + b \cos \phi = \sqrt{a^2 + b^2} \sin\left(\phi + \tan^{-1}\frac{b}{a}\right).$$

12. The force (inertia force) on the piston of an engine is given by the equation

$$F = F_0\left(\cos\theta + \frac{R}{L}\cos 2\,\theta\right),$$

where θ is the angle which the crank arm makes with a fixed direction, R the length of the crank arm, L the length of the connecting rod, and F_0 a constant (the ideal centrifugal force at the crank pin center). Plot the curve, showing the value of F for any given position of the crank, when $F_0 = 600$, $R = 15$, $L = 45$.

13. s, the distance of the piston of a steam engine from its extreme position corresponding to $\theta = 0$, is given approximately by the equation

$$s = R\left(1 - \cos\theta + \frac{R}{2\,L}\sin^2\theta\right),$$

where R, L and θ have the same meaning as in Example 12. Plot the curve showing the relation between s and θ, when $R = 10$ and $L = 20$.

137. The Logarithmic Curve, $y = \log_{10}x$.

With the aid of a table of common logarithms and a knowledge of the fundamental properties of logarithms, we find the following corresponding values of x and y:

$x =$ 1, 2,	3,	4,	5,	10,	100,	.,	.,	∞.
$y =$ 0, 0.30,	0.48,	0.60,	0.70,	1,	2,	.,	.,	∞.
$x =$ $\frac{1}{2}$,	$\frac{1}{3}$,	$\frac{1}{4}$,	$\frac{1}{5}$,	0.1,	0.01,	.,	.,	0.
$y =$ $-$ 0.30,	$-$ 0.48,	$-$ 0.60,	$-$ 0.70,	$-$ 1,	$-$ 2,	.,	.,	$- \infty$.

In any case, if $x = \frac{1}{n}$, $y = -\log n$.

These values enable us to plot the portion of the curve shown in Fig. 167. They also give us a definite idea of the shape of the curve beyond the limits of the figure. To the right the curve diverges more and more from the x-axis as x increases, below the x-axis the curve approaches the y-axis more and more nearly as x approaches 0; that is, the logarithmic curve $y = \log_{10}x$ has the y-axis for an asymptote.

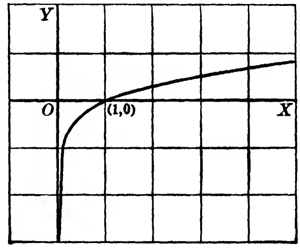


Fig. 167. $y = \log_{10} x$.

138. The Exponential Curve, $y = 10^x$.

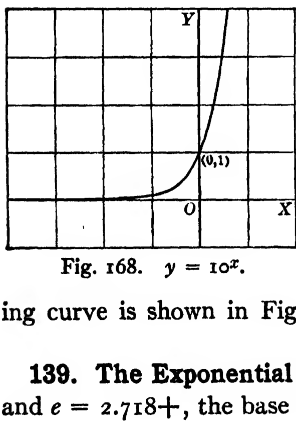


Fig. 168. $y = 10^x$.

Taking the common logarithm of each side of the equation $y = 10^x$, we obtain

$$\log_{10} y = x, \text{ or } x = \log_{10} y.$$

This shows that the exponential curve $y = 10^x$ may be obtained from the curve $y = \log_{10}x$ by interchanging the x's and y's, that is, by using the ordinates of the logarithmic curve for abscissas and the abscissas for ordinates. The resulting curve is shown in Fig. 168.

139. The Exponential Curves, $y = e^{kx}$, where k is any constant, and $e = 2.718+$, the base of the natural system of logarithms.

(a) $y = e^x$. Let us first consider the special case for $k = 1$.

A general idea of the shape of the curve $y = e^x$ may be gathered from the following sets of corresponding values of x and y:

$$\begin{array}{llllll}
x = 0, & \frac{1}{2}, & 1, & 2, & \cdot\cdot, & \infty, \\
y = 1, & (2.7)^{\frac{1}{2}} = 1.6, & 2.7, & (2.7)^2 = 7.4, & \cdot\cdot, & \infty, \\
x = -\frac{1}{2}, & & -1, & -2, & \cdot\cdot, & -\infty. \\
y = & \frac{1}{1.6} = 0.61, & \frac{1}{2.7} = 0.38, & \frac{1}{7.39} = 0.14, & \cdot\cdot, & -0.
\end{array}$$

From these values the curve may be sketched as in Fig. 168.

If the curve is to be plotted with greater accuracy than the above figures will permit, it is best to employ logarithms. For, on

taking the logarithm of each member of the equation $y = e^x$, we have

$$\log_{10} y = x \log_{10} e = x \log_{10} 2.718+ = 0.4343\, x.$$

Assigning to x in succession the values 0, 0.1, 0.2, ., ., 1, 2, 3, etc. we obtain by means of a table of logarithms,

$x =$ 0,	0.1,	0.2,	.,	.,	1,	2,	3, etc.,	
$\log y =$ 0,	0.04343,	0.08686,	.,	.,	0.4343,	0.8686,	1.3029, etc.,	
$y =$ 1,	1.1052,	1.2214,	.,	.,	2.718,	7.390,	20.085,	etc.

The intervals for x may be chosen as small as we please, and the table extended at will, depending on the accuracy desired and the extent of the region for which the curve is to be plotted. The values of y corresponding to negative values of x are the reciprocals of the values of y when x is positive.

(*b*) $y = e^{kx}$, k positive.

When $x = 0$, $y = 1$, therefore, no matter what value k has, the curve passes through the point (0, 1) Fig. 169.

Let $k > 1$. So long as x is positive, $e^x > 1$, and therefore $e^{kx} = (e^x)^k > e^x$, that is, to the right of the y-axis the curve $y = e^{kx}$, $k > 1$, lies above the curve $y = e^x$, and it will diverge from it the more, the greater the value of k. For negative values of x, $k > 1$, $e^{kx} = (e^x)^k < e^x$, that is, to the left of the y-axis the curve $y = e^{kx}$, $k > 1$, will lie between the curve $y = e^x$ and the x-axis and will converge the more rapidly to this axis the greater the value of k. For any given value of k the curve may be roughly sketched by means of a few properly chosen points. If greater accuracy is required, we first compute a table of values from the relation $\log_{10} y = 0.4343\, kx$.

Considering in like manner the cases when $k < 1$, we find that the curves $y = e^{kx}$, $k < 1$, lie between the curve $y = e^x$ and the straight line drawn parallel to the x-axis through the point (0, 1) and that the curve will approach this straight line more nearly the smaller the value of k. Fig. 169 shows the curves for the values $k = \frac{1}{3}$, $k = \frac{1}{2}$, $k = 1$, $k = 2$ and $k = 3$.

(*c*) $y = e^{kx}$, k negative, or $y = e^{-kx}$, k positive.

The curve $y = e^{-kx}$ (1) is most readily obtained from the curve $y = e^{kx}$ (2). For $y = e^{-kx} = (e^k)^{-x}$, from which it is plain that for a positive value of x, the ordinate y of (1) will be the same as the

value of y in (2) for the corresponding negative value of x, and vice versa, the y in (1) when x is negative will be equal to the y in (2) when x is positive. This means that the two curves $y = e^{-kx}$ and $y = e^{kx}$, k being the same in both cases, are symmetrical with respect to the y-axis, so that either one being given, the other may be traced from it without computing anew the coördinates of its points. Two such curves have the same relation to each other as an object and its reflection in a mirror. For this reason either of the two is said to be the reflection on the y-axis of the other. Fig. 169 shows the curves $y = e^{kx}$ for the values $k = -3, -2, -1, -\frac{1}{2}, -\frac{1}{3}$.

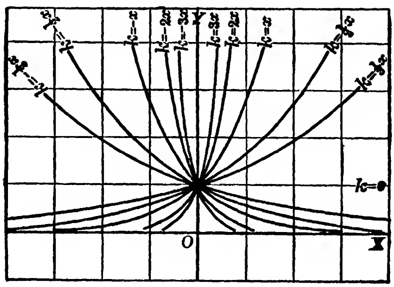


Fig. 169. $y = e^{kx}$.

140. The Compound Interest Law. In physics, chemistry and various branches of engineering, related quantities occur which are subject to laws which may be expressed by the formula

$$y = ae^{kx}, \tag{1}$$

or by the equivalent formula

$$x = \frac{1}{k}\log\frac{y}{a}. \tag{2}$$

In fact, it can be shown that whenever two quantities are so related, that the ratio between their changes is always proportional to one of the quantities, the relation between them may be expressed by either (1) or (2), where x and y are the quantities in question, and k and a constants which depend on the rate of change and the initial values of x and y. The amount of money due at any time on a sum of money put out at compound interest, the interest being added to the principal not at stated intervals but as fast as it accrues, varies with the time according to this law. For this reason the general law

expressed by (1) or (2) is commonly known as the *compound interest law*. The student of science will meet numerous examples of the compound interest law. A few simple examples are given in the set of problems which follows.

EXERCISE 60

Plot the following curves:

1. $y = \log_e x$.	2. $y = 10^{\frac{x}{2}}$.	3. $y = 10^{-x}$.
4. $y = 10^{-\frac{x}{2}}$.	5. $y = \frac{1}{2}\log_e \frac{x}{2}$.	6. $y = 3 \cdot 10^{\frac{x}{3}}$.
7. $y = 3^x$.	8. $y = 3^{-x}$.	9. $y = 2^{2x}$.
10. $y = 2^{-2x}$.	11. $y = 2\,e^x$	12. $y = -2\,e^x$.

13. Given the curve $y = e^{kx}$, trace the curve $y = -e^{kx}$ without computing the coördinates of its points.

(Suggestion. The required curve is the reflection on the x-axis of the given curve.)

14. The amount A due on a principal of P dollars, put out at compound interest at r per cent for t years, the interest being added to the principal as fast as it accrues, is given by the formula

$$A = Pe^{\frac{rt}{100}}.$$

Plot the curve showing the amount due at any given time, when $P = 100$ and $r = 5$.

(Suggestion. Use different values of t for abscissas and the corresponding values of A for ordinates.)

15. The work W due to the expansion of steam in the cylinder of a steam engine, while expanding from a given volume to a volume V, is given by the formula

$$W = a \log_e V - b,$$

where a and b are two constants depending upon the initial volume and pressure. Plot the curve showing the relation between V and W for any volume from $V = 1$ to $V = 4$, when $a = 15{,}000$, and $b = 0$.

16. Newton's law of cooling. The difference θ between the temperature of a body and the temperature of the medium surrounding it is given by the formula

$$\theta = be^{-at},$$

where a and b are constants depending upon the nature of the body and the initial temperatures. If when $t = 0$, $\theta = 50°$, and when $t = 60$ seconds, $\theta = 25°$, plot a curve showing the temperature θ at any time t up to 60 seconds.

141. The Catenary, $y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)$ in which e is the base of the natural system of logarithms and c any positive constant.

We first consider the special case $c = 1$, $y = \frac{1}{2}(e^x + e^{-x})$.

Put $y_1 = e^x$ and $y_2 = e^{-x}$, then $y = \frac{1}{2}(y_1 + y_2)$, which shows that any ordinate of the required curve is equal to half the sum of the corresponding ordinates of two curves $y_1 = e^x$ and $y_2 = e^{-x}$. If, therefore, these two curves are drawn first, as in Fig. 170, the

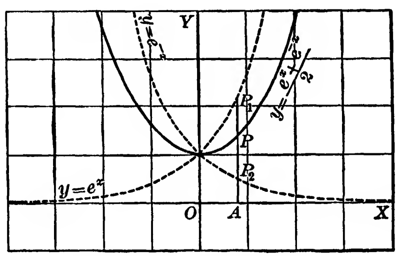


Fig. 170. The Catenary.

required curve is easily drawn by taking each ordinate, as AP, equal to half the sum of the ordinates AP_1 and AP_2.

It should be observed that

$$y = \frac{y_1 + y_2}{2} = y_2 + \frac{y_1 - y_2}{2} = AP_2 + \frac{P_2P_1}{2},$$

that is, each point P bisects the line joining two corresponding points P_1 and P_2.

We now pass to the general case. $y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)$ may be written

$$\frac{y}{c} = \frac{e^{\frac{x}{c}} + e^{-\frac{x}{c}}}{2} \qquad (1)$$

which becomes

$$y' = \frac{e^{x'} + e^{-x'}}{2}, \qquad (2)$$

where

$$y' = \frac{y}{c}, \qquad x' = \frac{x}{c}.$$

From this we see that if the coördinates x', y' of any point on the curve (2) be multiplied by c, the resulting numbers are the coördinates x, y of some point on the curve (1). The curve (1) is therefore merely some magnification of the curve (2). The curve in Fig. 170 may therefore be taken to represent any equation of the form (1), provided the proper scale is employed in the construction. It is only necessary to let each unit of length along the coördinate axis represent c units.

The curve $y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)$ is called the catenary. It is the curve formed by a rope or flexible cable suspended between two points. c is the ratio of the horizontal tension to the weight of a unit length of the rope or cable.

142. The Curve of Damped Vibrations, $y = ae^{-kx} \sin (cx + d)$. Put

$$y_1 = ae^{-kx}, \qquad (1), \qquad\qquad y_2 = \sin (cx + d), \qquad (2)$$

then

$$y = y_1 y_2 = ae^{-kx} \sin (cx + d), \qquad (3)$$

that is, any ordinate of the required curve is equal to the product of the corresponding ordinates of the two curves (1) and (2).

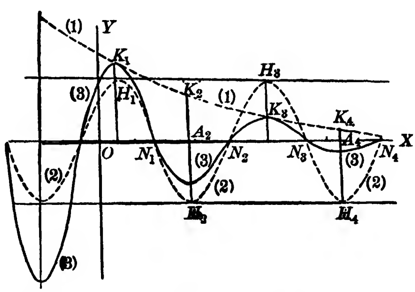


Fig. 171. The Curve of Damped Vibrations.

Let the curves (1) and (2) first be constructed separately, as in Fig. 171. Now observe:

(*a*) Since y_2 is less or at most equal to 1, y of (3) is less or at most equal to y_1, that is, the required curve lies below the curve (1).

(*b*) Whenever y_1 or y_2 equals zero, y of (3) equals zero also, hence the required curve crosses the x-axis at the points N_1, N_2, N_3, N_4, etc., at which the curve (2) crosses this axis.

(*c*) Whenever $y_2 = 1$, as at H_1, H_3, etc., y of (3) equals y_1, hence the required curve touches the curve (1) (since it cannot cross it) at the points K_1, K_3, etc.

(*d*) Whenever $y_2 = -1$, as at H_2, H_4, etc., $y = -y_1$. This enables us to locate the points K_2, K_4, of the required curve.

The shape of the curve (3) is now apparent. It is a wave curve of constant wave length, but the amplitude of the successive waves diminishes. The rapidity with which the amplitude decreases depends on the value of the constant k in (1). This constant k is known as the *logarithmic decrement* of the curve.

Like the exponential curve and the sine curve, this curve finds frequent applications in science. While the sine curve represents free vibrations, that is, vibrations not retarded by friction or otherwise, the curve $y = ae^{-kx} \sin (cx + d)$ represents damped vibrations, that is, vibrations suffering resistance of some kind. A pendulum vibrating in air or water, waves propagated in a viscous fluid and oscillatory discharges from an electric condenser, are familiar examples of damped vibratory motion.

EXERCISE 61

Plot the curves:

1. $y = e^{-x} \cos x$. 2. $y = e^{x} \sin x$. 3. $y = \dfrac{e^x - e^{-x}}{2}$.

4. $y = \dfrac{2}{e^x - e^{-x}}$. (Suggestion. The given equation may be written $y = \dfrac{1}{y'}$, where $y' = \dfrac{e^x - e^{-x}}{2}$.)

5. $y = \dfrac{2}{e^x + e^{-x}}$. See suggestion under Problem 4.

6. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$. (Suggestion. The given equation may be written $y = \dfrac{y_1}{y_2}$, where $y_1 = \dfrac{e^x - e^{-x}}{2}$, $y_2 = \dfrac{e^x + e^{-x}}{2}$.)

7. $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$. (Suggestion. This equation is the reciprocal of that in Problem 6.)

8. The six functions,

$$y_1 = \frac{e^x - e^{-x}}{2}, \qquad y_2 = \frac{e^x + e^{-x}}{2}, \qquad y_3 = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

$$y_4 = \frac{2}{e^x - e^{-x}}, \qquad y_5 = \frac{2}{e^x + e^{-x}}, \qquad y_6 = \frac{e^x + e^{-x}}{e^x - e^{-x}},$$

are known as the *hyperbolic functions*. Show that

$$y_2^2 - y_1^2 = 1,$$
$$y_5^2 + y_3^2 = 1,$$
$$y_6^2 - y_4^2 = 1.$$

Compare the graphs of these functions, Fig. 172. The graphs are numbered in correspondence with the suffixes of the y's in their equations above.

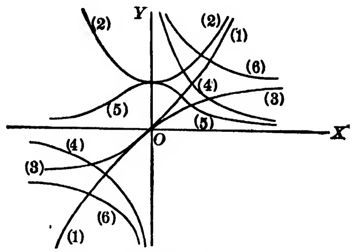


Fig. 172. The Six Hyperbolic Curves.

9. A cable, weighing one pound to the foot, is suspended between two piers under a tension of 100 pounds. Plot the curve which it forms.

10. The displacement of the end of a spring, from its position of equilibrium, at the time t seconds is given by the equation

$$s = ae^{-kt} \sin \frac{2\pi}{T} t,$$

where a is the amplitude of the vibration of the string if there were no friction, k represents the effect of friction retarding the vibration, and T is the time of a single vibration. Plot a curve showing the displacement at any moment during the first ten seconds, if

$$a = 5, \quad k = \tfrac{1}{2}, \quad T = 2.$$

CHAPTER XIV

TRIGONOMETRIC REPRESENTATION OF COMPLEX QUANTITIES

143. Imaginary Numbers. If we solve the equation

$$x^2 + 1 = 0,$$

we obtain $x = \pm\sqrt{-1}$. Similarly, every equation of the form

$$x^2 + a^2 = 0,$$

where a is any real number, has for its solution

$$x = \pm\sqrt{-a^2} = \pm a\sqrt{-1},$$

which contains $\sqrt{-1}$ as a factor.

This new number $\sqrt{-1}$, whose square is -1, is commonly denoted by the letter i and is called the *imaginary unit.*

Since $i^2 = -1$, it follows that

$$i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \quad i^6 = -1, \quad i^7 = -i, \quad i^8 = 1, \text{ etc.,}$$

that is,

Every integral power of i is equal to 1, i, -1, or $-i$, according as the exponent of the power when divided by 4 leaves the remainder 0, 1, 2, or 3.

In symbols

$$i^{4n} = 1, \qquad i^{4n+1} = i, \qquad i^{4n+2} = -1, \qquad i^{4n+3} = -i.$$

Numbers like i, $2i$, $-5i$, $\frac{i}{2}$, $\sqrt{a}\cdot i$, etc., are called imaginary numbers or quantities. Every imaginary number consists of the imaginary unit i multiplied by some real coefficient. From the rule for the powers of i it follows that every even power of an imaginary number is a real number, every odd power of an imaginary number is again an imaginary number.

.44. Geometrical Representation of Imaginary Numbers. Every positive or negative real number x may be represented geometrically by a distance on the x axis measured to the right or left according as x is positive or negative. Equal distances measured

in the same direction represent equal numbers, but equal distances measured in opposite directions represent numbers which are equal in magnitude but opposite in sign. Thus, in Fig. 173, if the points X_{-2}, X_{-1}, O, X_1, X_2 are taken at equal intervals, and OX_1 represents unity, then each of the segments

$$OX_1,\ X_1X_2,\ X_{-2}X_{-1},\ X_{-1}O \text{ represents } +1,$$

and each of the segments

$$OX_{-1},\ X_{-1}X_{-2},\ X_2X_1,\ X_1O \text{ represents } -1.$$

In general, if the segment PQ represents the number a, QP represents the number $-a$, and therefore $PQ + QP = a + (-a) = 0$. It follows that a line segment PQ on the x-axis will continue to represent the same real number if it is moved along the x-axis from one position to another, so long as its direction remains unchanged.

If all the line segments have the same initial point O, as OX_1, OX_2, OX_{-1}, OX_{-2}, Fig. 173, the terminal points X_1, X_2, X_{-1}, X_{-2} will represent the numbers quite as well as the segments themselves. Thus, if $OX_1 = 1$, the numbers $1, 2, -1, -2$ are represented equally well by the segments OX_1, OX_2, OX_{-1}, OX_{-2}, and by the points X_1, X_2, X_{-1}, X_{-2}, respectively.

Likewise every positive or negative imaginary number iy may be represented geometrically by a distance on the y-axis measured upwards or downwards according as y is positive or negative. Equal distances on the y-axis measured in the same direction represent equal imaginary numbers, equal distances on the y-axis measured in opposite directions represent imaginary numbers which are equal but opposite in sign. Thus, in Fig. 173, if the points Y_{-2}, Y_{-1}, O, Y_1, Y_2 are taken at equal intervals, and OY_1 is taken in length equal to OX_1, then each of the segments OY_1, Y_1Y_2, $Y_{-2}Y_{-1}$, $Y_{-1}O$ represents $+i$, and each of the segments OY_{-1}, $Y_{-1}Y_{-2}$, Y_2Y_1, Y_1O represents $-i$.

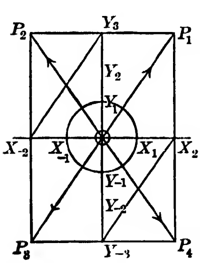


Fig. 173.

In general, if the segment RS represents the number ai, SR represents the number $-ai$, and therefore $RS + SR = ai + (-ai) = 0$. It follows that any line segment on the y-axis will continue to represent the same imaginary number if it is moved along the y-axis from one position to another so long as its direction remains unchanged.

If all the line segments have the same initial point O, as OY_1, OY_2, OY_{-1}, OY_{-2}, the terminal points Y_1, Y_2, Y_{-1}, Y_{-2} will represent the imaginary numbers quite as well as the segments themselves. Thus, if OY_1 represents i, the numbers i, $2i$, $-i$, $-2i$ are equally well represented by the segments OY_1, OY_2, OY_{-1}, OY_{-2}, and by the points Y_1, Y_2, Y_{-1}, Y_{-2}, respectively.

When the points on the x-axis and y-axis are used to represent real and imaginary numbers respectively, these axes are referred to as the *axis of reals* and the *axis of imaginaries* respectively.

The reason for choosing the axis of imaginaries at right angles to the axis of reals is found in the following considerations:

The equation $i^2 = -1$ may be put in the following form,

$$+1 : i = i : -1,$$

that is, i is a mean proportional between $+1$ and -1. Now the geometric construction for a mean proportional gives the perpendicular OY_1, erected at O, to the semicircle constructed on $X_{-1}X_1$ as a diameter. Hence, if OX_1 represents $+1$, and OX_{-1} represents -1, OY_1 will represent $\sqrt{+1 \times -1}$ or i.

Or, we may reason as follows: $i^2 = i \times i = -1$, that is, two successive multiplications by i have the same effect as multiplication by -1. Now multiplying any line segment OX_1 by -1 gives $-OX_1 = X_1O = OX_{-1}$, that is, multiplying by -1 has the effect of turning OX_1 through an angle of $180°$, hence multiplying by i should have the effect of turning OX_1 through half this angle or $90°$, that is, $i \times OX_1 = OY_1$, so that if OX_1 represents 1, OY_1 must represent i.

145. Geometrical Representation of Complex Numbers.

If we solve the equation $x^2 - 4x + 13 = 0$,

we obtain $$x = 2 \pm 3\sqrt{-1},$$

that is, $$x = 2 + 3i, \text{ or } 2 - 3i.$$

Each of these numbers consists of two parts, a real part $+2$, and an imaginary part $+3i$ or $-3i$. A number like $2 + 3i$, or $2 - 3i$, which consists of two parts, one which is real and the other imaginary, is called a *complex number*. The general form of a complex number is $a + bi$, where a and b are real numbers. a and b may be positive or negative, integral or fractional, rational or irrational.

A complex number $a + bi$ is represented geometrically by a line segment joining the origin O to the point whose coördinates are a and b. Thus, the complex number $2 + 3i$ is represented by the line segment OP_1, Fig. 173, obtained by joining the origin O to the point P_1, whose coördinates are 2, 3. Here also the direction of the line segment as well as its length is to be considered. The line-segments OP_1, OP_2, OP_3, OP_4 are equal in length, but they do not represent the same numbers. OP_1 represents the number $2 + 3i$, OP_2 represents $-2 + 3i$, OP_3 represents $-2 - 3i$, and OP_4 represents $2 - 3i$.

Two directed line segments which have the same length and the same direction are considered equal, and may be taken to represent the same number. It follows that a line segment will continue to represent the same number if it is moved parallel to itself. Thus, OP_1, $X_{-2}Y_3$, P_3O, $Y_{-3}X_2$ being parallel and equal in length, all represent the same complex number $2 + 3i$.

If the directed line segments all have the same initial point O, as OP_1, OP_2, OP_3, OP_4, the terminal points P_1, P_2, P_3, P_4 may be taken to represent the complex numbers as well as the directed segments. In this sense we may speak of the points P_1, P_2, P_3, P_4 as representing the numbers $2 + 3i$, $-2 + 3i$, $-2 - 3i$, $2 - 3i$, respectively.

If in $a + bi$ we put $b = 0$, we get all possible real numbers a; if $a = 0$, we get all possible imaginary numbers bi, so that the complex numbers $a + bi$ include as special cases all real and imaginary numbers.

146. Trigonometric Representation of Complex Numbers.

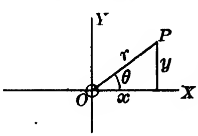


Fig. 174.

Let P (Fig. 174) represent the point $x + iy$. Let r denote the length OP and θ the angle which OP makes with the x-axis. Then

$$x = r\cos\theta, \qquad y = r\sin\theta, \tag{1}$$

and

$$x + iy = r(\cos\theta + i\sin\theta). \tag{2}$$

$r(\cos\theta + i\sin\theta)$ is called the *trigonometric form* of the complex number $x + iy$; r, which is always positive, being the distance of the point P from the origin θ, is called the *modulus* or *absolute value*, and θ is called the *argument* or *amplitude* of the complex number $x + iy$.

From (1) we obtain

$$\cos\theta = \frac{x}{r},\quad \sin\theta = \frac{y}{r},\quad \tan\theta = \frac{y}{x},\quad r = \sqrt{x^2+y^2}. \tag{3}$$

The equations (3) enable us to express any complex number in the trigonometric form. Thus, if $x + iy = 3 + 4i$, we find

$$r = \sqrt{3^2+4^2} = 5,\quad \cos\theta = \tfrac{3}{5},\quad \sin\theta = \tfrac{4}{5},\quad \tan\theta = \tfrac{4}{3},$$

so that

$$3 + 4i = 5\,(\cos 53^\circ 8' + i \sin 53^\circ 8').$$

If $x + iy = 3 - 4i$, $x = 3$, $y = -4$, and we have

$$r = \sqrt{3^2 + (-4)^2} = 5,\quad \cos\theta = \tfrac{3}{5},\quad \sin\theta = -\tfrac{4}{5},\quad \tan\theta = -\tfrac{4}{3},$$

from which $\theta = 306^\circ 52'$ or $-53^\circ 8'$,

so that $3 - 4i = 5\,(\cos 306^\circ 52' + i \sin 306^\circ 52')$,

or

$$\begin{aligned} 3 - 4i &= 5\,(\cos -53^\circ 8' + i \sin -53^\circ 8'),\\ &= 5\,(\cos 53^\circ 8' - i \sin 53^\circ 8'). \end{aligned}$$

Conversely, if a complex number is given in the trigonometric form, equations (1) enable us to express it in the form $x + iy$. Thus, if the given number is $2\,(\cos 30^\circ + i \sin 30^\circ)$, $r = 2$, $\theta = 30^\circ$, and we have from (1)

$$x = 2\cos 30^\circ = \sqrt{3},\quad y = 2 \sin 30^\circ = 1,$$

so that

$$2\,(\cos 30^\circ + i \sin 30^\circ) = \sqrt{3} + i.$$

147. Geometric Addition and Subtraction of Complex Numbers.

Let OP and OP' (Fig. 175) represent any two complex numbers $x + iy$ and $x' + iy'$ respectively. Complete the parallelogram $OPQP'$, having OP and OP' for two of its sides. Draw QC perpendicular and PD parallel to OX. The right triangles PDQ and OBP' are equal (Why?), hence

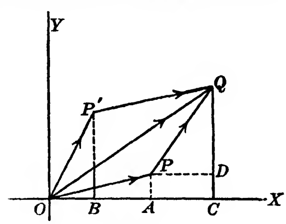


Fig. 175.

$$PD = OB = x',\qquad DQ = BP' = y',$$

and therefore

$$OC = OA + PD = x + x',\ CQ = AP + DQ = y + y',$$

so that the directed line OQ represents the complex number

$$(x + x') + i\,(y + y') = (x + iy) + (x' + iy').$$

The sum of two complex numbers represented by OP and OP′ respectively is represented by the diagonal drawn from O of the parallelogram formed with OP and OP′ as sides; or,

To find geometrically the sum of two complex numbers represented by OP and OP′ respectively, move OP′ parallel to itself so that its initial point falls on the terminal point P of OP. The directed line, drawn to connect the initial point O of OP to the terminal point of OP′ in its new position, represents the required sum.

In its second form this rule may be easily extended to the sum of any number of complex numbers. To add geometrically the numbers represented by OP, OP_1, OP_2, OP_3, etc., we first add any two of them, as OP and OP_1; to their sum we add any third, as OP_2; to the sum of these three we add a fourth, as OP_3, etc. Leaving out the lines which are not needed to obtain the final result, we obtain the following rule:

Construct a broken line $OPQRS$ . . . such that OP coincides with OP, PQ is equal and parallel to OP_1, QR is equal and parallel to OP_2, RS is equal and parallel to OP_3, etc. The directed line joining O to the terminal point of this broken line represents the required sum.

The length of the line representing the sum of two or more complex numbers will be less, or at most equal, to the sum of the lengths of the several lines representing the numbers added, hence

The modulus of the sum of any number of complex numbers is less than, or at most equal to, the sum of their moduli.

Geometric subtraction follows from geometric addition. Let OQ and OP (Fig. 175) represent any two complex numbers whose difference $OQ - OP$ is to be found geometrically. Since $OP + OP' = OQ$, $OQ - OP = OP' = PQ$. PQ, or its equal OP', represents then the required difference, in words,—

The difference OQ — OP between two complex numbers, represented by OQ and OP respectively, is represented by the third side PQ of the triangle of which the other two sides are OP and OQ.

148. Application of Complex Quantities to Physics. It is shown in physics that if OP and OP' (Fig. 175) represent in length two forces acting in the directions indicated by the arrows, then OQ, the diagonal of the parallelogram of which OP and OP' form the sides, will represent their resultant, both as regards magnitude and

direction. It follows that if OQ represents a given force in magnitude and direction, and OP its component in the direction OP, $PQ = OP'$ must represent the other component. The laws of composition and resolution of forces are then precisely those which govern the geometrical addition and subtraction of complex numbers; we may therefore compound or resolve forces by adding or subtracting the complex numbers which represent them.

Suppose n forces $f_1, f_2, f_3, \ldots f_n$ (Fig. 176) act in the same plane on the same point P with the respective intensities $r_1, r_2, r_3, \ldots r_n$ pounds, and at angles $\theta_1, \theta_2, \theta_3, \ldots \theta_n$ with the x-axis. The forces are then represented by the complex numbers: —

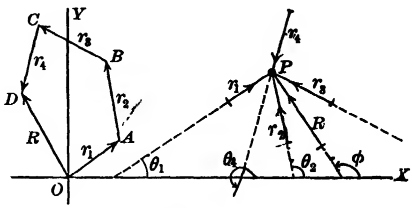


Fig. 176.

$$f_1 = r_1 (\cos\theta_1 + i \sin\theta_1),$$
$$f_2 = r_2 (\cos\theta_2 + i \sin\theta_2),$$
$$f_3 = r_3 (\cos\theta_3 + i \sin\theta_3),$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$f_n = r_n (\cos\theta_n + i \sin\theta_n).$$

Adding all these quantities we obtain for the resultant:

$$\begin{aligned} F &= r_1 \cos\theta_1 + r_2 \cos\theta_2 + r_3 \cos\theta_3 + \ldots + r_n \cos\theta \\ &\qquad + i (r_1 \sin\theta_1 + r_2 \sin\theta_2 + r_3 \sin\theta_3 + \ldots + r_n \sin\theta_n) \\ &= R (\cos\phi + i \sin\phi), \end{aligned}$$

where R is the modulus and ϕ the argument of the complex numbers representing the resultant force F.

The resultant may be found geometrically by constructing OA equal and parallel to r_1, AB equal and parallel to r_2, BC equal and parallel to r_3, and so on. Then OD, the directed line joining the origin O to the terminal extremity of the broken line thus constructed, represents the resultant force F both as regards magnitude and direction.

We will apply the method just explained to the solution of Problem 16, Exercise 26. Taking AP for the direction of the x-axis we have

$r_1 = 15,\ r_2 = 6,\qquad r_3 = 5.7,\qquad r_4 = 7.9,\qquad r_5 = 12.3,\qquad r_6 = 10.$

$\theta_1 = 0°,\ \theta_2 = 12°\,30',\ \theta_3 = 31°\,21',\ \theta_4 = 47°\,46',\ \theta_5 = 58°\,10',\ \theta_6 = 72°\,18'$

$$\begin{aligned}
f_1 &= 15\,(\cos 0° + i \sin 0°) &&= 15.0000 + i\ \ 0.0000\\
f_2 &= 6\,(\cos 12°\,30' + i \sin 12°\,30') &&= 5.8578 + i\ \ 1.2984\\
f_3 &= 5.7\,(\cos 31°\,21' + i \sin 31°\,21') &&= 4.8678 + i\ \ 2.9657\\
f_4 &= 7.9\,(\cos 47°\,46' + i \sin 47°\,46') &&= 5.3104 + i\ \ 5.8492\\
f_5 &= 12.3\,(\cos 58°\,10' + i \sin 58°\,10') &&= 6.4882 + i\ 10.4501\\
f_6 &= 10\,(\cos 72°\,18' + i \sin 72°\,18') &&= 3.0400 + i\ \ 9.5270
\end{aligned}$$

Adding

$$F = R\,(\cos\phi + i \sin\phi) = 40.5642 + i\,30.0904$$

$$R = \sqrt{\overline{40.5642}^2 + \overline{30.0904}^2} = 50.51,$$

$$\phi = \cos^{-1}\frac{40.56}{50.51} = \sin^{-1}\frac{30.09}{50.51} = 36°\,34',$$

whence $$F = 50.51\,(\cos 36°\,34' + i \sin 36°\,34'),$$

that is, the resultant force is 50.51 pounds and acts at an angle 36° 34′ with AP.

149. Historical Note. The method of representing complex numbers by points in a plane is often referred to as Argand's representation, after J. R. Argand, a French mathematician, who gave a discussion of the method in 1806. This is another misnomer, for it is now known that Caspar Wessel, a German, published the same method as early as 1799. The method was, however, completely forgotten until in 1831 it was rediscovered and applied by the great Gauss, to whom is generally conceded the credit of having established the true theory of imaginary numbers. To Gauss we owe the term "complex numbers" and the symbol i to represent the imaginary unit. The term "imaginary" was first used by Descartes. The choice of the term was unfortunate, for imaginary numbers as now understood are no more "imaginary" in the ordinary meaning of that term than are negative numbers. In view of their geometrical interpretation the name "lateral numbers" has been suggested in the place of the name "imaginary." The trigonometric form of complex numbers was first used by the great French mathematician Cauchy. The concept of complex numbers and their geometrical representation forms the basis of many of the branches o' higher mathematics, and is indispensable as well to the study of theoretical physics.

EXERCISE 62

1. Represent geometrically each of the numbers,

$$2, -3, 4i, -i, 1+i, 1-i, 2+3i, -\tfrac{1}{2}-i.$$

2. Represent geometrically each of the numbers,

$$2(\cos 30^\circ + i \sin 30^\circ),$$

$\cos 60^\circ + i \sin 60^\circ$, $3(\cos 120^\circ + i \sin 120^\circ)$, $\frac{1}{3}(\cos 240^\circ + i \sin 240^\circ)$, $\cos 0^\circ + i \sin 0^\circ$, $\cos 90^\circ + i \sin 90^\circ$, $\sqrt{2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)$.

3. Express the following numbers in the trigonometric form:

$\frac{1}{2}\sqrt{2} + i\frac{1}{2}\sqrt{2}$, $\frac{1}{2} + i\frac{1}{2}\sqrt{3}$, $\frac{1}{2} - i\frac{1}{2}\sqrt{3}$, $\frac{1}{2}\sqrt{3} + \frac{1}{2}i$, $-\frac{1}{2}\sqrt{3} + \frac{1}{2}i$, i, -1, 1.

Ans. $\cos 45^\circ + i \sin 45^\circ$, $\cos 60^\circ + i \sin 60^\circ$, . . . , $\cos 0^\circ + i \sin 0^\circ$.

4. Express each of the numbers in Problem 2 in the form $x + iy$.

Ans. $\sqrt{3} + i$, $\frac{1}{2} + i\frac{1}{2}\sqrt{3}$, . . . , $1 + i$.

5. Compare the moduli of the following numbers:

$\sqrt{3} + i$, $\sqrt{3} - i$, $-\sqrt{3} + i$, $-\sqrt{3} - i$, $\sqrt{2} + i\sqrt{2}$, $\sqrt{2} - i\sqrt{2}$, $-\sqrt{2} + i\sqrt{2}$, $-\sqrt{2} - i\sqrt{2}$, $1 + i\sqrt{3}$, $1 - i\sqrt{3}$, $-1 + i\sqrt{3}$, $-1 - i\sqrt{3}$, 2, $2i$, -2, $-2i$.

What conclusion can you draw with reference to the location of the points representing these numbers?

6. Add geometrically $1 + i$ and $2 + i$; $1 + 2i$, $2 + i$, and $1 - i$; $1 + 2i$, $-2 + 3i$, $1 - i$, $-3 - i$, and $3 - 3i$; 1, i, 2, and $1 - i$.

7. Add $1 + 2i$, $2 + i$, and $1 - i$ geometrically, in three different ways:

(*a*) By adding the first and second and then the third,
(*b*) By adding the first and third and then the second,
(*c*) By adding the second and third and then the first.

8. Subtract geometrically $2 + 3i$ from $3 + 4i$; $2 - i$ from $1 + i$; $3 + 2i$ from $1 - i$.

9. Prove the rule for geometric subtraction separately by means of the relation

$$(x + iy) - (x' + iy') = (x - x') + i(y - y').$$

10. Prove that the sum of the complex numbers representing the sides of a polygon taken in order equals zero.

11. Three forces of 151, 106, and 61 horse power respectively, make angles of $50° 04' 30''$, $211° 20' 305''$, and $-96°$ respectively with the x-axis Show that the resultant is zero, that is, that the forces are in equilibrium.

150. Multiplication and Division of Complex Numbers. Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1), \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

represent any two complex numbers. Their product is

$$\begin{aligned} z_1z_2 &= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= \boldsymbol{r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]}. \qquad (1) \end{aligned}$$

Dividing z_1 by z_2 we obtain

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}\cdot\frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2}\cdot\frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}, \\ &= \frac{r_1}{r_2}\,\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{(\cos^2\theta - i^2\sin^2\theta) = 1}, \\ &= \boldsymbol{\frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]}. \qquad (2) \end{aligned}$$

From equations (1) and (2) it appears that,

The product of two complex numbers is another complex number whose modulus is the product of the moduli and whose argument is the sum of the arguments of the numbers.

The quotient of two complex numbers is another complex number whose modulus is the modulus of the dividend divided by the modulus of the divisor, and whose argument is the argument of the dividend diminished by the argument of the divisor.

Corollary 1. Since

$$\begin{aligned} 1 &= \cos 0° + i\sin 0°, \\ i &= \cos 90° + i\sin 90°, \\ -1 &= \cos 180° + i\sin 180°, \end{aligned}$$

the modulus in each case being unity, it follows that:

multiplying any complex number by 1 leaves it unchanged,
multiplying any complex number by i increases its argument by $90°$,
multiplying any complex number by -1 increases its argument by $180°$.

In the first case the line segment representing the complex number is left unchanged, in the second case the line segment is turned in the positive direction through an angle of 90°, in the third case the line segment is reversed.

Corollary 2. $(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = 1$; therefore $\cos\theta + i\sin\theta$ and $\cos\theta - i\sin\theta$ are reciprocals, and in general the reciprocal of $r(\cos\theta + i\sin\theta)$ is $\frac{1}{r}(\cos\theta - i\sin\theta)$.

Two numbers such as $r(\cos\theta + i\sin\theta)$ and $r(\cos\theta - i\sin\theta)$ are said to be *conjugate* to each other. Each is called the *conjugate* of the other.

151. Powers of Complex Numbers. By (1), Article 150, we have

$$\begin{aligned} z_1z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]. \end{aligned}$$

Let us multiply this result by a third complex number z_3, thus:

$$\begin{aligned} z_1z_2z_3 &= r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)] \times r_3(\cos\theta_3 + i\sin\theta_3) \\ &= r_1r_2r_3[\cos(\theta_1+\theta_2+\theta_3) + i\sin(\theta_1+\theta_2+\theta_3)]. \end{aligned}$$

Similarly we obtain for the product of n factors

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1),\ z_2 = r_2(\cos\theta_2 + i\sin\theta_2),\ \ldots\ z_n = r_n(\cos\theta_n + i\sin\theta_n)$$

$$z_1z_2\ldots z_n = r_1r_2\ldots r_n[\cos(\theta_1+\theta_2+\ldots+\theta_n) + i\sin(\theta_1+\theta_2+\ldots+\theta_n)] \quad (1)$$

From (1) we see that,

The modulus of the product of any number of complex numbers is equal to the product of the moduli of the factors, and the argument of the product is equal to the sum of the arguments of the factors.

Now let us suppose that the n factors in (1) are all equal, each factor being

$$z = r(\cos\theta + i\sin\theta),$$

we then have

$$z^n = r^n(\cos n\theta + i\sin n\theta). \quad (2)$$

In particular, if $r = 1$,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \quad (3)$$

Equation (3) embodies one of the most famous theorems of modern analysis. It is known as *DeMoivre's* theorem*, after its discoverer, and may be stated thus:

The argument of the nth power of any complex number is equal to n times the argument of the number. The theorem may be shown to hold for any value of n, negative, fractional, irrational or even imaginary, but unless n is integral $\cos n\theta + i \sin n\theta$ represents but one of the several values which $(\cos\theta + i\sin\theta)^n$ may have.

To illustrate the use of DeMoivre's theorem, we will employ it to raise $\frac{1}{2} + \frac{i}{2}\sqrt{3}$ to the 9th power. Changing $\frac{1}{2} + \frac{i}{2}\sqrt{3}$ to the trigonometric form we have

$$r = 1, \quad \cos\theta = \frac{1}{2}, \quad \sin\theta = \frac{\sqrt{3}}{2}, \quad \text{hence} \quad \theta = \frac{\pi}{3},$$

and

$$\left(\frac{1}{2} + \frac{i}{2}\sqrt{3}\right)^9 = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^9 = \cos\frac{9\pi}{3} + i\sin\frac{9\pi}{3} = -1.$$

Similarly

$$(1+i)^5 = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^5 = 4\sqrt{2}\left[\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right]$$
$$= -4(1+i).$$

152. Roots of Complex Numbers. Suppose it is required to find the nth root of the complex number $z = r(\cos\theta + i\sin\theta)$. Denote the root $\sqrt[n]{z}$ by

$$z' = r'(\cos\theta' + i\sin\theta').$$

Then $(z')^n = z$, and we have by DeMoivre's theorem

$$[r'(\cos\theta' + i\sin\theta')]^n = r'^n(\cos n\theta' + i\sin n\theta') = r(\cos\theta + i\sin\theta),$$

from which

$$r' = \sqrt[n]{r}, \quad n\theta' = \theta, \theta + 2\pi, \theta + 4\pi, \theta + 6\pi, \ldots \theta + 2k\pi,$$

since when an angle is increased by any number of times 2π both the sine and cosine remain unchanged. It follows that

* DeMoivre (1667–1754) created a large part of that portion of trigonometry which deals with complex numbers. His death has a curious psychological interest. Shortly before his death he slept a little longer each day, until when the limit of twenty-four hours was reached, he died in his sleep.

$$r' = \sqrt[n]{r},\ \theta' = \frac{\theta}{n},\ \frac{\theta + 2\pi}{n},\ \frac{\theta + 4\pi}{n},\ \frac{\theta + 6\pi}{n},\ \ldots\ \frac{\theta + k\pi}{n}$$

and

$$\begin{aligned} z' = \sqrt[n]{z} &= r^{\frac{1}{n}}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right) \\ &= r^{\frac{1}{n}}\left(\cos\frac{\theta + 2\pi}{n} + i\sin\frac{\theta + 2\pi}{n}\right) \\ &= r^{\frac{1}{n}}\left(\cos\frac{\theta + 4\pi}{n} + i\sin\frac{\theta + 4\pi}{n}\right) \\ &\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot \\ &= r^{\frac{1}{n}}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right), \end{aligned}$$

where k is any integer.

These values are not all different, for when $k = n$ we have

$$\begin{aligned} r^{n}\left(\frac{\cos\theta + 2n\pi}{n} + i\sin\frac{\theta + 2n\pi}{n}\right) &= r^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n} + 2\pi\right) + i\sin\left(\frac{\theta}{n} + 2\pi\right)\right] \\ &= r^{\frac{1}{n}}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right), \end{aligned}$$

and similarly

$$r^{\frac{1}{n}}\left(\cos\frac{\theta+2(n+1)\pi}{n} + i\sin\frac{\theta+2(n+1)\pi}{n}\right) = r^{\frac{1}{n}}\left(\cos\frac{\theta+2\pi}{n} + i\sin\frac{\theta+2\pi}{n}\right),$$

and generally

$$\begin{aligned} r^{\frac{1}{n}}&\left(\cos\frac{\theta + 2(n + \mu)\pi}{n} + i\sin\frac{\theta + 2(n + \mu)\pi}{n}\right) \\ &= r^{\frac{1}{n}}\left(\cos\frac{\theta + 2\mu\pi}{n} + \sin\frac{\theta + 2\mu\pi}{n}\right), \end{aligned}$$

so that z' has only n distinct values corresponding to the values

$$k = 0,\ 1,\ 2,\ \ldots\ n - 1,$$

that is,

Every complex number $r\,(\cos\theta + i\sin\theta)$ has n nth roots given by the formula

$$[r(\cos\theta + i\sin\theta)]^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right], \qquad (1)$$

where k has the values $0, 1, 2, \ldots n - 1$.

EXAMPLE 1 To find the fourth roots of 1.

Solution. In the trigonometric form $1 = \cos 0^\circ + i \sin 0^\circ$, hence by the formula (1) above

$$\sqrt[4]{1} = \left(\cos\left(\frac{0 + 2k\pi}{4} + i \sin \frac{0 + 2k\pi}{4}\right), \text{ where } k = 0, 1, 2, \text{ or } 3.$$

Denoting the four roots by z_0, z_1, z_2, z_3 respectively, we have

$$z_0 = \cos\frac{0}{4} + i \sin\frac{0}{4} = 1, \qquad z_1 = \cos\frac{2\pi}{4} + i \sin\frac{2\pi}{4} = i,$$

$$z_2 = \cos\frac{4\pi}{4} + i \sin\frac{4\pi}{4} = -1, \qquad z_3 = \cos\frac{6\pi}{4} + i \sin\frac{6\pi}{4} = -i.$$

EXAMPLE 2. To find the fifth roots of -1.

Solution. In the trigonometric form $-1 = \cos\pi + i \sin\pi$, hence

$$\sqrt[5]{-1} = \cos\frac{\pi + 2k\pi}{5} + i \sin\frac{\pi + 2k\pi}{5}, \text{ where } k = 0, 1, 2, 3, 4.$$

This gives the five values

$$z_0 = \cos\frac{\pi}{5} + i \sin\frac{\pi}{5},\ z_1 = \cos\frac{3\pi}{5} + i \sin\frac{3\pi}{5},\ z_2 = \cos\pi + i \sin\pi,$$

$$z_3 = \cos\frac{7\pi}{5} + i \sin\frac{7\pi}{5},\ z_4 = \cos\frac{9\pi}{5} + i \sin\frac{9\pi}{5}.$$

The last two roots may be written equally well

$$z_3 = \cos\frac{3\pi}{5} - i \sin\frac{3\pi}{5},\ z_4 = \cos\frac{\pi}{5} - i \sin\frac{\pi}{5},$$

for

$$\cos\frac{7\pi}{5} = \cos\left(2\pi - \frac{3\pi}{5}\right) = \cos\frac{3\pi}{5},\ \sin\frac{7\pi}{5} = \sin\left(2\pi - \frac{3\pi}{5}\right) = -\sin\frac{3\pi}{5},$$

and

$$\cos\frac{9\pi}{5} = \cos\left(2\pi - \frac{\pi}{5}\right) = \cos\frac{\pi}{5},\ \sin\frac{9\pi}{5} = \sin\left(2\pi - \frac{\pi}{5}\right) = -\sin\frac{\pi}{5},$$

so that finally

$$\sqrt[5]{-1} = \cos\frac{\pi}{5} \pm i \sin\frac{\pi}{5},\ \cos\frac{3\pi}{5} \pm i \sin\frac{3\pi}{5},\ \cos\pi + i \sin\pi = -1.$$

153. To Solve the Equation $z^n - 1 = 0$. If $z^n - 1 = 0$, then

$$z^n = 1 = \cos 0 + i \sin 0,$$

and the n roots are

$$z_0 = \cos \frac{0}{n} + i \sin \frac{0}{n} = 1,$$

$$z_1 = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n},$$

$$z_2 = \cos \frac{4\pi}{n} + i \sin \frac{4\pi}{n},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$z_{n-2} = \cos \frac{2(n-2)\pi}{n} + i \sin \frac{2(n-2)\pi}{n},$$

$$z_{n-1} = \cos \frac{2(n-1)\pi}{n} + i \sin \frac{2(n-1)\pi}{n}.$$

Now

$$\cos \frac{2(n-1)\pi}{n} = \cos\left(2\pi - \frac{2\pi}{n}\right) = \cos \frac{2\pi}{n},$$

$$\sin \frac{2(n-1)\pi}{n} = \sin\left(2\pi - \frac{2\pi}{n}\right) = -\sin \frac{2\pi}{n},$$

hence

$$z_{n-1} = \cos \frac{2\pi}{n} - i \sin \frac{2\pi}{n}, \text{ and similarly } z_{n-2} = \cos \frac{4\pi}{n} - i \sin \frac{4\pi}{n},$$

that is, z_1 and z_{n-1} are conjugate complex numbers, and likewise z_2 and z_{n-2}, z_3 and z_{n-3}, etc., are each pairs of conjugate numbers.

(*a*) Let $n = 2m + 1$, an odd number. Then besides the first root z_0 there are m pairs of conjugate roots, that is, the roots are

$$1,\ \cos \frac{2\pi}{n} \pm i \sin \frac{2\pi}{n},\ \cos \frac{4\pi}{n} \pm i \sin \frac{4\pi}{n},\ \ldots\ \cos \frac{2m\pi}{n} \pm i \sin \frac{2m\pi}{n}.$$

(*b*) Let $n = 2m$, an even number. Then

$$z_n = \cos \frac{2m\pi}{n} + i \sin \frac{2m\pi}{n} = \cos \pi + i \sin \pi = -1,$$

and the roots are

$$1,\ \cos \frac{2\pi}{n} \pm i \sin \frac{2\pi}{n},\ \cos \frac{4\pi}{n} \pm i \sin \frac{4\pi}{n},\ \ldots,$$

$$\cos \frac{(m-1)2\pi}{n} \pm i \sin \frac{(m-1)2\pi}{n},\ -1.$$

The roots of the equation $z^n - 1 = 0$ are represented geometrically by the n lines (or by their terminal points) drawn from the origin as center to the circumference of a circle, radius unity, so as to divide the circumference into n equal parts, one of these lines coinciding with the positive direction of the x-axis.

154. To Solve the Equation $z^n + 1 = 0$. If $z^n + 1 = 0$, then

$$z^n = -1 = \cos\pi + i\sin\pi,$$

hence the n roots are

$$z_0 = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n},$$

$$z_1 = \cos\frac{3\pi}{n} + i\sin\frac{3\pi}{n},$$

$$z_2 = \cos\frac{5\pi}{n} + i\sin\frac{5\pi}{n},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$z_{n-2} = \cos\frac{(2n-3)\pi}{n} + i\sin\frac{(2n-3)\pi}{n},$$

$$z_{n-1} = \cos\frac{(2n-1)\pi}{n} + i\sin\frac{(2n-1)\pi}{n},$$

where z_0 and z_{n-1}, z_1 and z_{n-2}, etc., are pairs of conjugate roots.

(*a*) Let $n = 2m + 1$, an odd number. The middle root is

$$z_m = \cos\frac{(2m+1)\pi}{n} + i\sin\frac{(2m+1)\pi}{n} = \cos\pi + i\sin\pi = -1,$$

so that the n roots are

$$\cos\frac{\pi}{n} \pm i\sin\frac{\pi}{n},\quad \cos\frac{3\pi}{n} \pm i\sin\frac{3\pi}{n},\ \ldots,$$

$$\cos\frac{(2m-1)\pi}{n} \pm i\sin\frac{(2m-1)\pi}{n},\quad -1.$$

(*b*) Let $n = 2m$, an even number. Then the roots are

$$\cos\frac{\pi}{n} \pm i\sin\frac{\pi}{n},\quad \cos\frac{3\pi}{n} \pm i\sin\frac{3\pi}{n},\ \ldots,$$

$$\cos\frac{(2m-1)\pi}{n} \pm i\sin\frac{(2m-1)\pi}{n}.$$

The roots of the equation $z^n + 1 = 0$ are represented geometrically by the n lines (or by their terminal points) drawn from the origin as center to the circumference of a circle, radius unity, so as to divide the circumference into n equal parts, the positive x-axis being taken to bisect the angle between a pair of consecutive lines.

Exercise 63

Compute the following expressions by DeMoivre's theorem, then verify your results by expanding the binomials by the binomial theorem.

1. $(1 + i)^2$. *Ans.* $2i$.

2. $(1 - i)^4$. *Ans.* -4.

3. $\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^3$.

4. $\left(\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}\right)^2$. *Ans.* $-i$.

5. $\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$.

6. Show that each of the following expressions equals -1,

$$\left(\cos\frac{\pi}{5} \pm i\sin\frac{\pi}{5}\right)^5, \quad \left(\cos\frac{3\pi}{5} \pm i\sin\frac{3\pi}{5}\right)^5, \quad (\cos\pi \pm i\sin\pi)^5.$$

7. Construct the points representing the expressions

$$\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right), \quad \left(\cos\frac{\pi}{5} - i\sin\frac{\pi}{5}\right), \text{ etc., in Problem 6.}$$

8. Find the seven seventh roots of 1 and plot the points representing these roots.

Ans. $1, \cos\frac{2\pi}{7} \pm i\sin\frac{2\pi}{7}, \quad \cos\frac{4\pi}{7} \pm i\sin\frac{4\pi}{7}, \quad \cos\frac{6\pi}{7} \pm i\sin\frac{6\pi}{7}$.

9. Find the seven seventh roots of -1 and plot the points representing them.

10. If $z_0, z_1, z_2, \ldots z_{n-1}$ are the roots of the equation $z^n - 1 = 0$, show that

$$z_2 = z_1^2, \; z_3 = z_1^3, \; z_4 = z_1^4, \; \ldots \; z_{n-1} = z_1^{n-1}.$$

11. Find all the values of $\sqrt[5]{1 + i}$ and represent them geometrically. *Ans.* $\sqrt[10]{2}\,(\cos 9° + i\sin 9°)$, $\sqrt[10]{2}\,(\cos 81° + i\sin 81°)$, etc.

155. The Cube Roots of Unity. Let u_0, u_1, u_2 represent the three cube roots of 1, then by the preceding article

$$u_0 = \cos 0 + i \sin 0 = 1,$$

$$u_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \frac{-1}{2} + i\frac{\sqrt{3}}{2},$$

$$u_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \frac{-1}{2} - i\frac{\sqrt{3}}{2}.$$

Now

$$u_1^2 = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = u_2,$$

$$u_2^2 = \left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)^2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = u_1,$$

also

$$u_1 u_2 = u_1^3 = u_2^3 = 1,$$

that is,

The square of either of the imaginary cube roots of unity equals the other, and their product equals unity.

We may then denote the cube roots of unity by

$$1,\ \omega,\ \omega^2,$$

where ω is either one of the roots u_1, u_2.

156. The Cube Roots of Any Real or Complex Number. Let $r(\cos\theta + i\sin\theta)$ represent any number and z_0, z_1, z_2 its cube roots.

We have

$$\left.\begin{aligned}
z_0 &= r^{\frac{1}{3}}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right),\\
z_1 &= r^{\frac{1}{3}}\left(\cos\frac{\theta+2\pi}{3} + i\sin\frac{\theta+2\pi}{3}\right)\\
&= r^{\frac{1}{3}}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right)\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = \omega z_0,\\
z_2 &= r^{\frac{1}{3}}\left(\cos\frac{\theta+4\pi}{3} + i\sin\frac{\theta+4\pi}{3}\right)\\
&= r^{\frac{1}{3}}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right)\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = \omega^2 z_0,
\end{aligned}\right\} \qquad (1)$$

where ω has the meaning given it in Article 155.

Furthermore, by applying the results just obtained

$$\omega z_1 = \omega^2 z_0 = z_2, \qquad \omega^2 z_1 = \omega^3 z_0 = z_0, \tag{2}$$

$$\omega z_2 = \omega^3 z_0 = z_0, \qquad \omega^2 z_2 = \omega^4 z_0 = \omega z_0 = z_1. \tag{3}$$

From (1), (2) and (3) it appears that any two cube roots of a number may be obtained by multiplying the third by ω and ω^2 respectively. Thus:

from (1), $z_1 = \omega z_0$, $z_2 = \omega^2 z_0$,

from (2) $z_2 = \omega z_1$, $z_0 = \omega^2 z_1$,

from (3), $z_0 = \omega z_2$, $z_1 = \omega^2 z_2$,

where $\omega = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.

157. Solution of Cubic Equations. Every cubic equation can be expressed in the form

$$a_0x^3 + 3\,a_1x^2 + 3\,a_2x + a_3 = 0, \tag{1}$$

where a_0, a_1, a_2, a_3 are given numbers.

Equation (1) can be replaced by another in which the second term is missing by putting

$$x = \frac{z - a_1}{a_0}. \tag{2}$$

On making this substitution, equation (1) reduces to

$$z^3 + 3\,Hz + G = 0, \tag{3}$$

where

$$H = a_0a_2 - a_1^2,\ G = a_0^2a_3 - 3\,a_0a_1a_2 + 2\,a_1^3. \tag{4}$$

Now put

$$z = u + v, \tag{5}$$

then (3) becomes

$$(u + v)^3 + 3\,H(u + v) + G = u^3 + v^3 + 3(u + v)(H + uv) + G = 0. \tag{6}$$

If furthermore we put

$$H + uv = 0, \quad \text{that is,} \quad uv = -H, \tag{7}$$

then (6) becomes

$$u^3 + v^3 = -G, \tag{8}$$

and substituting for v in (8) its value from (7)

$$u^3 - \frac{H^3}{u^3} = -G, \text{ or } u^6 + Gu^3 - H^3 = 0. \tag{9}$$

(9) is a quadratic equation in u^3. Solving

$$u^3 = \frac{-G \pm \sqrt{G^2 + 4H^3}}{2},$$

$$v^3 = -G - u^3 = \frac{-G \mp \sqrt{G^2 + 4H^3}}{2},$$

so that for either sign we obtain

$$z = u + v = \sqrt{\frac{-G + \sqrt{G^2 + 4H^3}}{2}} + \sqrt{\frac{-G - \sqrt{G^2 + 4H^3}}{2}}. \quad (10)$$

From (7) $v = \frac{-H}{u}$, so that (10) may be written $z = u - \frac{H}{u}$,

where $$u = \sqrt{\frac{-G + \sqrt{G^2 + 4H^3}}{2}}.$$

But the cube root of every number has three values, which by Article 156 may be denoted by u, ωu, $\omega^2 u$ respectively, where u is any one of these roots. The three values of z which satisfy the equation (3) are therefore

$$\begin{aligned} z_0 &= u - \frac{H}{u}, \\ z_1 &= \omega u - \frac{H}{\omega u} = \omega u - \frac{\omega^2 H}{u}, \\ z_2 &= \omega^2 u - \frac{H}{\omega^2 u} = \omega^2 u - \frac{\omega H}{u}, \end{aligned} \quad (11)$$

u being either one of the cube roots of $\frac{-G + \sqrt{G^2 + 4H^3}}{2}$. (12)

In applying this method to the solution of any equation of the form (1), we first find H and G from (4), then u from (12), then the three values of z from (11) and finally the three corresponding values of x from (2).

EXAMPLE 1. Solve the equation $8x^3 + 12x^2 - 42x - 95 = 0$.

Solution. Here $a_0 = 8$, $a_1 = 4$, $a_2 = -14$, $a_3 = -95$.

From (4),

$$H = a_0 a_2 - a_1^2 = -128 = -2^7,$$
$$G = a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3 = -4608 = -2^9 3^2.$$

From (12),

$$\sqrt{G^2 + 4H^3} = \sqrt{2^{18}\,3^4 - 4.2^{21}} = 7.2^9,$$

$$u = \sqrt[3]{\frac{-G + \sqrt{G^2 + 4H^3}}{2}} = \sqrt[3]{\frac{2^9 3^2 + 7.2^9}{2}} = 2^4.$$

From (11),

$$z_0 = u - \frac{H}{u} = 2^4 + 2^3 = 24,$$

$$z_1 = \omega u - \frac{\omega^2 H}{u} = 16\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + 8\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -12 + 4i\sqrt{3},$$

$$z_2 = \omega^2 u - \frac{\omega H}{u} = 16\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) + 8\left(-\frac{1}{2} = i\frac{\sqrt{3}}{2}\right) = -12 - 4i\sqrt{3}.$$

From (2)

$$x_0 = \frac{z_0 - a_1}{a_0} = \frac{24 - 4}{8} = 2.5,$$

$$x_1 = \frac{z_1 - a_1}{a_0} = \frac{-12 + 4i\sqrt{3} - 4}{8} = -2 + \frac{\sqrt{3}}{2}i,$$

$$x_2 = \frac{z_2 - a_1}{a_0} = \frac{-12 - 4i\sqrt{3} - 4}{8} = -2 - \frac{\sqrt{3}}{2}i.$$

Check. $$\left(x + 2 - \frac{\sqrt{3}}{2}i\right)\left(x + 2 + \frac{\sqrt{3}}{2}i\right) = x^2 + 4x + \tfrac{19}{4} = 0,$$

$$(x^2 + 4x + \tfrac{19}{4})(x - \tfrac{5}{2}) = x^3 + \tfrac{3}{2}x^2 - \tfrac{21}{4}x - \tfrac{95}{8} = 0,$$

or $$8x^3 + 12x^2 - 42x - 95 = 0.$$

158. The Irreducible Case. When G^2+4H^3 is positive, as in the preceding example, its square root is real, and u, which is the cube root of $\frac{-G + \sqrt{G^2 + 4H^3}}{2}$ can be found by the rules of arithmetic. But if $G^2 + 4H^3$ is negative, its square root will be imaginary, and we must employ DeMoivre's theorem to find u. This is the so-called "irreducible case" of the cubic equation. Its solution is as follows:

Since $G^2 + 4H^3$ is negative, $-(G^2 + 4H^3)$ will be positive, and we may put

$$u^3 = \frac{-G + i\sqrt{-(G^2 + 4H^3)}}{2} = r(\cos\theta + i\sin\theta),$$

whence, by Article 146, (3),

$$r = \sqrt{\frac{(-G)^2 + [-(G^2 + 4H^3)]}{4}} = \sqrt{-H^3}, \tag{1}$$

$$\cos\theta = \frac{-G}{2\sqrt{-H^3}}. \tag{2}$$

$$u = r^{\frac{1}{3}}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right) = \sqrt{-H}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right),$$

$$\frac{-H}{u} = \sqrt{-H}\left(\cos\frac{\theta}{3} - i\sin\frac{\theta}{3}\right),$$

$$\omega u = \sqrt{-H}\left(\cos\frac{\theta + 2\pi}{3} + i\sin\frac{\theta + 2\pi}{3}\right),$$

$$\frac{-H}{\omega u} = \sqrt{-H}\left(\cos\frac{\theta + 2\pi}{3} - i\sin\frac{\theta + 2\pi}{3}\right),$$

$$\omega^2 u = \sqrt{-H}\left(\cos\frac{\theta + 4\pi}{3} + i\sin\frac{\theta + 4\pi}{3}\right),$$

$$\frac{-H}{\omega^2 u} = \sqrt{-H}\left(\cos\frac{\theta + 4\pi}{3} - i\sin\frac{\theta + 4\pi}{3}\right).$$

Hence,

$$\begin{aligned} z_0 &= u - \frac{H}{u} = 2\sqrt{-H}\cos\frac{\theta}{3}, \\ z_1 &= \omega u - \frac{H}{\omega u} = 2\sqrt{-H}\cos\frac{\theta + 2\pi}{3}, \\ z_2 &= \omega^2 u - \frac{H}{\omega^2 u} = 2\sqrt{-H}\cos\frac{\theta + 4\pi}{3}, \end{aligned} \tag{3}$$

and from Article 157, (2),

$$x = \frac{z - a_1}{a_0}. \tag{4}$$

EXAMPLE 1. Solve the equation $3x^3 + 3x^2 - 3x - 2 = 0$.

Solution. $a_0 = 3$, $a_1 = 1$, $a_2 = -1$, $a_3 = -2$.

$H = a_0^2 a_2 - a_1^2 = -4$, $G = a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3 = -7$,

$G^2 + 4H^3 = -207$, hence we are dealing with an irreducible case.

From (2),

$$\cos\theta = \frac{-G}{2\sqrt{-H^3}} = \frac{7}{16} = 0.4375, \quad \theta = 64^\circ\, 3'\, 20''.$$

From (3),

$$z_0 = 2\sqrt{-H}\cos\frac{\theta}{3} = 4\cos 21^\circ\, 21'\, 7'' = 3.7256,$$

$$z_1 = 2\sqrt{-H}\cos\frac{\theta+2\pi}{3} = 4\cos 141^\circ\, 21'\, 7'' = -3.1241,$$

$$z = 2\sqrt{-H}\cos\frac{\theta+4\pi}{3} = 4\cos 261^\circ\, 21'\, 7'' = -0.6015.$$

From (4),

$$x_0 = \frac{z_0 - a_1}{a_0} = 0.9085, \qquad x_1 = \frac{z_1 - a_1}{a_0} = -1.3747, \qquad x_2 = \frac{z_2 - a_1}{a_0} = -0.5338.$$

Exercise 64

1. $\omega_1 = \dfrac{-1 + i\sqrt{3}}{2}, \quad \omega_2 = \dfrac{-1 - i\sqrt{3}}{2}.$

By actual multiplication, show that

$$\omega_1^2 = \omega_2, \quad \omega_2^2 = \omega_1, \quad \omega_1\omega_2 = 1, \quad \omega_1^3 = 1, \quad \omega_2^3 = 1.$$

Show also that $1 + \omega_1 + \omega_2 = 0$.

2. Compute all the cube roots of 27; of $27\,i$.

Ans. $3, \dfrac{-3+3i\sqrt{3}}{2}, \dfrac{-3-3i\sqrt{3}}{2};$
$\dfrac{3\sqrt{3}+3i}{2}, \dfrac{-3\sqrt{3}+3i}{2}, -3i.$

3. Compute all the cube roots of $1 + i$.

Ans. $\sqrt[6]{2}\left(\cos\dfrac{\pi}{12} + i\sin\dfrac{\pi}{12}\right), \sqrt[6]{2}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right),$
$\sqrt[6]{2}\left(\cos\dfrac{11\pi}{12} + i\sin\dfrac{11\pi}{12}\right).$

Solve the following equations:

4. $x^3 - 6x - 9 = 0.$ *Ans.* $3, \dfrac{-3 \pm i\sqrt{3}}{2}.$

5. $4x^3 - 24x^2 + 45x - 25 = 0.$ *Ans.* 1, 2.5, 2.5.

6. $x^3 - 12x - 10 = 0.$ *Ans.* 3.8232, − 2.9304, − 0.8928.

7. $x^3 + 9x^2 + 24x + 19 = 0.$
Ans. By use of 4-place tables, −1.4680, −4.8794, −2.6527.

8. A man invests \$5000 and two years later \$2000 more. The interest was added to the principal at the end of each year. At the end of the third year the total increase was found to be \$2058.16. Find the rate per cent of profit. *Ans.* 11 %.

(Suggestion. Let $x = 1 + \text{rate}$.)

9. The inside of a tank is 3 ft. wider than it is deep and 3 ft. longer than it is wide. Its volume is 100 cu. ft. Find the dimensions of the tank. *Ans.* 2.284, 5.284, 8.284.

10. A loan of \$500 is to be repaid in three equal annual payments of \$190 each without further interest. Find the rate per cent.

11. In determining the deflection of a beam, uniformly loaded and supported at its two ends and points of trisection, the following equation occurs:

$$20x^3 - 24x^2 + 3 = 0.$$

Find its roots. *Ans.* 0.4460, 1.0687, − 0.3147.

159. To Express sin $n\theta$ and cos $n\theta$ in Terms of sin θ and cos θ. DeMoivre's theorem enables us to express the sine and cosine of any multiple angle $n\theta$ in terms of powers of the sine and cosine of θ. We need only compare separately the real parts and the imaginary parts of

$$\cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n \tag{1}$$

after expanding the right-hand member by the binomial theorem.

For shortness sake let us put

$$\cos\theta = c, \quad \sin\theta = s,$$

the right-hand side of (1) then becomes

$$\begin{aligned}(c+is)^n = c^n + nic^{n-1}s &+ \frac{n(n-1)}{1\cdot 2}i^2c^{n-2}s^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}i^3c^{n-3}s^3 \\ &+ \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}i^4c^{n-4}s^4 + \text{etc.,} \\ = c^n + nic^{n-1}s &- \frac{n(n-1)}{1\cdot 2}c^{n-2}s^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}ic^{n-3}s^3 \\ &+ \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}c^{n-4}s^4 + \text{etc.,}\end{aligned}$$

since $\qquad i^2 = -1,\ i^3 = -i,\ i^4 = 1$, etc.

The real part of this expression must equal $\cos n\theta$, since this is the real part of the equivalent left-hand member in (1), and for a like reason the imaginary part must equal $\sin n\theta$, hence we have

$$\cos n\theta = c^n - \frac{n(n-1)}{1\cdot 2}c^{n-2}s^2 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}c^{n-4}s^4 - \text{etc.} \qquad (2)$$

$$\sin n\theta = nc^{n-1}s - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}c^{n-3}s^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}c^{n-5}s^5 - \text{etc.} \qquad (3)$$

Thus, if $n = 2$,

$$\cos 2\theta = c^2 - \frac{2(2-1)}{1\cdot 2}c^0 s^2 = c^2 - s^2 = \cos^2\theta - \sin^2\theta,$$

$$\sin 2\theta = 2cs = 2\sin\theta\cos\theta.$$

If $n = 3$,

$$\cos 3\theta = c^3 - \frac{3(3-1)}{1\cdot 2}cs^2 = c^3 - 3cs^2 = \cos^3\theta - 3\cos\theta\sin^2\theta$$
$$= 4\cos^3\theta - 3\cos\theta,$$

$$\sin 3\theta = 3c^2s - \frac{3(3-1)(3-2)}{1\cdot 2\cdot 3}c^0 s^3 = 3\cos^2\theta\sin\theta - \sin^3\theta$$
$$= 3\sin\theta - 4\sin^3\theta,$$

results which agree with those obtained in Exercise 50, Problems 10 and 11.

160. To Express $\cos\theta$ and $\sin\theta$ in Terms of Sines and Cosines of Multiple Angles. By Article 150, cor. 2,

$$\cos\theta + i\sin\theta \text{ and } \cos\theta - i\sin\theta \text{ are reciprocals.}$$

Put $\quad z = \cos\theta + i\sin\theta, \qquad$ then $\quad \dfrac{1}{z} = \cos\theta - i\sin\theta,$

$$z^n = \cos n\theta + i\sin n\theta, \qquad \frac{1}{z^n} = \cos n\theta - i\sin n\theta,$$

hence

$$z + \frac{1}{z} = 2\cos\theta, \qquad z - \frac{1}{z} = 2i\sin\theta,$$

$$z^n + \frac{1}{z^n} = 2\cos n\theta, \qquad z^n - \frac{1}{z^n} = 2i\sin n\theta.$$

Now let us expand $\left(z+\frac{1}{z}\right)^n$ and $\left(z-\frac{1}{z}\right)^n$ by the binomial theorem,

$$\left(z+\frac{1}{z}\right)^n = 2^n\cos^n\theta = z^n + nz^{n-2} + \frac{n(n-1)}{1\cdot 2}z^{n-4} + \cdots$$
$$+\frac{n(n-1)}{1\cdot 2}\cdot\frac{1}{z^{n-4}} + n\frac{1}{z^{n-2}} + \frac{1}{z^n}, \qquad (1)$$

$$\left(z-\frac{1}{z}\right)^n = 2^n i^n\sin^n\theta = z^n - nz^{n-2} + \frac{n(n-1)}{1\cdot 2}z^{n-4} - \cdots$$
$$\pm\frac{n(n-1)}{1\cdot 2}\cdot\frac{1}{z^{n-4}} \mp n\frac{1}{z^{n-2}} \pm \frac{1}{z^n}, \qquad (2)$$

where in (2) the upper signs in the last terms are to be used when n is even and the lower signs when n is odd.

Let us group together the first and last term in each of the expressions (1) and (2), also the second and second last, the third and third last, etc.; we may then write

$$2^n\cos^n\theta = \left(z^n+\frac{1}{z^n}\right) + n\left(z^{n-2}+\frac{1}{z^{n-2}}\right)$$
$$+\frac{n(n-1)}{1\cdot 2}\left(z^{n-4}+\frac{1}{z^{n-4}}\right) + \cdots \qquad (3)$$

$$2^n i^n\sin^n\theta = \left(z^n \pm \frac{1}{z^n}\right) - n\left(z^{n-2}\pm\frac{1}{z^{n-2}}\right)$$
$$+\frac{n(n-1)}{1\cdot 2}\left(z^{n-4}\pm\frac{1}{z^{n-4}}\right) + \cdots \qquad (4)$$

where in (4) the upper signs are to be used when n is even and the lower signs when n is odd. The total number of terms in each binomial expression is one more than the index n, and since we have grouped the terms in pairs, there will be one term left over in case n is even. This term will not contain z at all, for since the exponent of z diminishes by 2 for each successive term, it will be 0, and $z^0 = 1$.

Let us now substitute for $z^n+\frac{1}{z^n}$, $z^n-\frac{1}{z^n}$, etc., their values $2\cos n\theta$, $2i\sin n\theta$, etc., and divide out the common factor 2. This gives

$$2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta$$
$$+\frac{n(n-1)}{1\cdot 2}\cos(n-4)\theta + \text{etc.}, \qquad (5)$$

$$2^{n-1} i^n \sin^n \theta = \cos n\theta - n\cos(n-2)\theta + \frac{n(n-1)}{1\cdot 2}\cos(n-4)\theta - \text{etc.,} \quad (n \text{ even}) \tag{6}$$

$$= i\left[\sin n\theta - n\sin(n-2)\theta + \frac{n(n-1)}{1\cdot 2}\sin(n-4)\theta - \text{etc.,}\right] \quad (n \text{ odd}). \tag{6'}$$

The factor i disappears in either case, for when n is even, say $2m$, we have $i^n = i^{2m} = (-1)^m$, which is $+1$ or -1 according as m is even or odd, and when n is odd, say $2m+1$, we can divide both sides of the equation (6′) by i and have left on the left side $i^{2m} = +1$ or -1 as before.

EXAMPLES. If $n = 2$,

$$2\cos^2\theta = \cos 2\theta + 1, \quad \text{or} \quad \cos^2\theta = \frac{1+\cos 2\theta}{2}.$$

$$2i^2\sin^2\theta = \cos 2\theta - 1, \quad \text{or} \quad \sin^2\theta = \frac{1-\sin 2\theta}{2}.$$

If $n = 3$,

$$2^2\cos^3\theta = \cos 3\theta + 3\cos\theta, \quad \cos^3\theta = \frac{3\cos\theta + \cos 3\theta}{4}.$$

$$2^2 i^3\sin^3\theta = i(\sin 3\theta - 3\sin\theta), \quad \sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}.$$

If $n = 4$,

$$2^3\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 6, \quad \cos^4\theta = \frac{6 + 4\cos 2\theta + \cos 4\theta}{8}.$$

$$2^3 i^4\sin^4\theta = \cos 4\theta - 4\cos 2\theta + 6, \quad \sin^4\theta = \frac{6 - 4\cos 2\theta + \cos 4\theta}{8}.$$

EXERCISE 65

1. By the method of Article 159 express $\cos 4\theta$ and $\sin 4\theta$ each in powers of $\sin\theta$ and $\cos\theta$.

Ans. $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$,
$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$.

2. Show that

$$\tan 4\theta = \frac{4\tan\theta(1-\tan^2\theta)}{1 - 6\tan^2\theta + \tan^4\theta}.$$

3. Show that

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta.$$
$$= \cos\theta\,(16\cos^4\theta - 20\cos^2\theta + 5).$$
$$\sin 5\theta = \sin^5\theta - 10\sin^3\theta\cos^2\theta + 5\sin\theta\cos^4\theta.$$
$$= \sin\theta\,(16\sin^4\theta - 20\sin^2\theta + 5).$$
$$\tan 5\theta = \frac{\tan\theta\,(\tan^4\theta - 10\tan^2\theta + 5)}{5\tan^4\theta - 10\tan^2\theta + 1}.$$

4. By the method of Article 160 express $\cos\theta$ and $\sin\theta$ in terms of functions of multiple angles.

$$\cos^5\theta = \frac{20\cos\theta + 5\cos 3\theta + \cos 5\theta}{16}.$$
$$\sin^5\theta = \frac{20\sin\theta - 5\sin 3\theta + \sin 5\theta}{16}.$$

5. Show that

$$\cos^6\theta = \frac{20 + 15\cos 2\theta + 6\cos 4\theta + \cos 6\theta}{32}.$$
$$\sin^6\theta = \frac{20 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta}{32}.$$

CHAPTER XV

TRIGONOMETRIC SERIES AND THE CONSTRUCTION OF TABLES

161. Definition of Infinite Series. An infinite series is an indicated sum of an endless number of terms. Infinite series are of common occurrence in arithmetic and algebra. Thus in

$$\frac{1}{9} = 0.11111 \ldots = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \ldots, \tag{1}$$

the right-hand member is an infinite series. Similarly, every recurring decimal may be expressed as an infinite series. Again, if we divide 1 by $1 - x$, according to the rule for long division we obtain

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + x^4 + \ldots \tag{2}$$

The right-hand member of this is an infinite series. The square root of every number which is not a perfect square and the cube root of every number which is not a perfect cube may be expressed in the form of an infinite series.

Since it is impossible to write down all the terms of an infinite series, the series is not completely determined until we know the law or rule according to which the various terms are formed. When this law is known we can write down any term that is needed. In the first series above, the law for the general term is $\frac{1}{10^n}$, and the series is completely expressed thus :

$$\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots + \frac{1}{10^n} + \ldots$$

In like manner the series (2) is completely expressed by

$$1 + x + x^2 + \ldots + x^{n-1} + \ldots$$

In each case n stands for the number of the term.

When the law of formation is clearly apparent from the first few terms, as in the above examples, the general term is not always expressed.

A series is frequently expressed by writing the Greek letter Σ before the general term; thus the second series above may be written $\Sigma\, x^{n-1}$, which is read "summation of x^{n-1}."

The general expression for an infinite series is

$$\Sigma\, u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

where $u_1, u_2, u_3, \ldots u_n$, are the terms of the series.

162. Convergent and Non-Convergent Series. Let S_n stand for the first n terms of the series, thus

$$\begin{aligned} S_2 &= u_1 + u_2, \\ S_3 &= u_1 + u_2 + u_3, \\ S_4 &= u_1 + u_2 + u_3 + u_4, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ S_n &= u_1 + u_2 + u_3 + \cdots + u_n. \end{aligned}$$

As n increases indefinitely (approaches ∞), one of three things must happen, —

(*a*) S_n may approach some finite quantity as its limit.

(*b*) S_n may become larger than every assignable finite quantity.

(*c*) S_n may neither approach a finite limit nor become infinite, but fluctuate between two or more different values.

No other alternative is conceivable. In the first case the series is said to be *convergent*, in the second *divergent*, in the third *oscillating*. Divergent and oscillating series are grouped together under the term *non-convergent* series.

If a series contains a variable, as the series (2), Article 161, the series may be convergent for certain values of the variable and non-convergent for other values.

EXAMPLE 1. Consider the series

$$x + x^2 + x^3 + \cdots + x^n + \cdots$$

(*a*) If $x = \frac{1}{2}$, the series becomes

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \cdots$$

$$S_1 = \frac{1}{2}, \quad S_2 = \frac{1}{2} + \frac{1}{4} = 1 - \frac{1}{4}, \quad S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1 - \frac{1}{8},$$

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}.$$

As n approaches ∞, $\frac{1}{2^n}$ approaches 0 and S_n approaches 1, hence the series comes under (*a*) and is convergent.

(*b*) Next let us put $x = 2$, then the series becomes

$$2 + 2^2 + 2^3 + \cdots + 2^n + \cdots$$

$$S_1 = 2, \quad S_2 = 2 + 4 = 6, \quad S_3 = 2 + 4 + 8 = 14,$$
$$S_n = 2 + 4 + 8 + \cdots + 2^n.$$

Plainly, as n approaches ∞, S_n approaches ∞, hence the series comes under (*b*) and is divergent.

(*c*) Finally, if $x = -1$, the series becomes

$$-1 + 1 - 1 + \cdots + (-1)^n + \cdots$$

$S_1 = -1$, $S_2 = 0$, $S_3 = -1$, $S_4 = 0$, $S_n = -1$ or 0 according as n is odd or even. In this case S_n neither approaches a finite limit nor becomes infinite, hence the series is oscillating.

Divergent series frequently lead to absurdities, and must therefore be avoided. For instance, if in the series (2), Article 161, we put $x = 2$, the left member becomes $\frac{1}{1-2} = -1$, while the right member becomes

$$1 + 2 + 4 + 8 + \cdots;$$

hence for the value $x = 2$, which makes the series divergent, the two members of (2) can no longer be considered equal.

163. Absolutely Convergent Series. A series which remains convergent after all its terms are made positive is said to be *absolutely convergent.*

Convergent series which become divergent when all the terms are made positive are said to be *semi-convergent* or *conditionally convergent.*

For instance, if in (2), Article 161, we put $x = -\frac{1}{2}$, the series becomes

$$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \cdots,$$

which is convergent, for it remains convergent when all its terms are taken with the positive sign. The series is therefore absolutely convergent.

The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is convergent, but when all the terms are made positive the resulting series is divergent. The given series is therefore a semi-convergent or conditionally convergent series.

Absolutely convergent series are subject to all the fundamental laws of algebra,* that is, they may be added, subtracted, multiplied and divided, like expressions consisting of a finite number of terms. This is not true of semi-convergent and divergent series. Curious results may be arrived at if this is not kept in mind. For example, take the divergent series

$$S = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + \cdots$$
$$0 = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

Adding
$$\begin{aligned} S &= 2 + 2 + 6 + 6 + 10 + 10 + 14 + 14 + \cdots \\ &= 4 + 12 + 20 + 28 + \cdots \\ &= 4\,(1 + 3 + 5 + 7 + \cdots) \\ &= 4\,S, \text{ which of course is absurd.} \end{aligned}$$

Or take the semi-convergent series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

which is known to be equal to $\log 2 = 0.69315 \cdots$ We may write

$$\begin{aligned} S &= 1 + \frac{1}{2} - \frac{2}{2} + \frac{1}{3} + \frac{1}{4} - \frac{2}{4} + \frac{1}{5} + \frac{1}{6} - \frac{2}{6} + \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots - \left(\frac{2}{2} + \frac{2}{4} + \frac{2}{6} + \frac{2}{8} + \cdots\right) \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots - \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right) \\ &= 0, \end{aligned}$$

that is, a constant $0.69312 \cdots$ equal to zero, which is absurd. These examples show that we cannot treat an infinite series like we do other expressions until we know whether it is absolutely convergent or not.

* For the proof of this statement we must refer the student to textbooks on higher algebra, such as Chrystal's Algebra, Chapter XXVI.

A series which is absolutely convergent will of course remain convergent when some or even all of the terms are made negative, but if a series is divergent when all its terms are positive it may become convergent when a certain proportion of its terms are made negative.

164. The Sum of an Infinite Series. When a series converges, not otherwise, the limit which S_n approaches as n approaches ∞ is called the sum of the series. Thus when we say that the sum of the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$$

is 1, we mean that the limit of

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

is 1 as n approaches ∞. A divergent series has no sum in the proper sense of that word.

165. The Limit of r^n as n Approaches ∞.

(*a*) Let $r = 1$. 1 multiplied by 1 equals 1 no matter how often the multiplication is repeated. Hence the limit of $r^n = 1$.

(*b*) Let $r > 1$. We may write $r = 1 + d$, where d is some positive quantity. By applying the binomial theorem we have

$$r^n = (1 + d)^n = 1 + nd + \cdots,$$

Now no matter how small d may be, if n is taken sufficiently large nd can be made larger than any assignable quantity, hence we see that as n approaches ∞, r^n approaches ∞ also.

(*c*) Let $r < 1$. Then $\frac{1}{r} > 1$, and since by (*a*) and (*b*) $\left(\frac{1}{r}\right)^n = \frac{1}{r^n}$ approaches ∞ as n approaches ∞, therefore r^n approaches $\frac{1}{\infty} = 0$ as n approaches ∞.

As n approaches ∞, r^n approaches 0, 1 or ∞ according as r is less than, equal to or greater than 1.

166. The Infinite Geometrical Series. Let

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1},$$

then

$$rS_n = \phantom{a +{}} ar + ar^2 + \cdots + ar^{n-1} + ar^n.$$

Subtracting $\qquad S_n - rS_n = (1 - r) S_n = a - ar^n = a(1 - r^n),$

from which $\qquad S_n = \dfrac{a(1 - r^n)}{1 - r}.$

(*a*) Let r be numerically less than 1. By Article 165 (*c*) as n approaches ∞ r^n approaches 0, hence S_n approaches the limit $\dfrac{a}{1 - r}$ and the series is convergent. Since the series converges when all its terms are positive, it is absolutely convergent.

(*b*) Let $r = 1$. Then $S_n = a + a + a + \cdot \cdot \cdot + a = na$, which approaches ∞ as n approaches ∞; hence in this case the series is divergent.

(*c*) Let $r = -1$. Then $S_n = a - a + a - \cdot \cdot \cdot + (-1)^{n-1} a = 0$ or a, according as n is even or odd. In this case the series is oscillating.

(*d*) Let r be numerically greater than 1. By Article 165 (*b*) as n approaches ∞ r^n approaches ∞, hence $S_n = \dfrac{a(1 - r^n)}{1 - r}$ approaches ∞ and the series is divergent.

The results may be summed up in the following theorem:

An infinite geometrical series is absolutely convergent if its ratio r is numerically less than 1, non-convergent if its ratio is numerically equal to or greater than 1. When convergent its sum

$$S = \frac{a}{1 - r},$$

where a is the first term and r the ratio.

167. Convergency Test. Let

$$u_1 + u_2 + u_3 + \cdot \cdot \cdot + u_n + \cdot \cdot \cdot$$

represent any infinite series of positive terms. Let r_n represent the ratio of any term u_n to the preceding term u_{n-1}. Then

$$u_2 = u_1 r_2, \quad u_3 = u_2 r_3, \quad u_4 = u_3 r_4, \quad \cdot \cdot \cdot \; u_n = u_{n-1} r_n, \quad \cdot \cdot \cdot$$

and consequently

$$\begin{aligned}
u_1 &= u_1, \\
u_2 &= u_1 r_2, \\
u_3 &= u_2 r_3 = u_1 r_2 r_3, \\
u_4 &= u_3 r_4 = u_1 r_2 r_3 r_4, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
u_n &= u_{n-1} r_n = u_1 r_2 r_3 r_4 \cdot \cdot \cdot r_n, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}$$

Adding $u_1 + u_2 + u_3 + u_4 + \cdots = u_1(1 + r_2 + r_2r_3 + r_2r_3r_4 + \cdots)$ (1)

$$< u_1(1 + R + R^2 + R^3 + \cdots) \quad (2)$$

$$> u_1(1 + r + r^2 + r^3 + \cdots) \quad (3)$$

where R is greater than the greatest, and r less than the least, of all the ratios $r_2, r_3, r_4, \ldots r_n, \ldots$

Now (2) is an infinite geometrical series which is convergent provided R is less than 1, and (3) is an infinite geometrical series which is divergent provided r is equal to or greater than 1, hence the infinite series

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

is convergent provided R is less than 1, divergent provided r is equal to or greater than 1.

The ratio $r_n = \dfrac{u_n}{u_{n-1}}$ is called the *ratio of convergency* or *test ratio* of the series Σu_n. We have then the following theorems:

A series is absolutely convergent if the test ratio is always less than some number R which is itself less than 1.

A series is divergent if the test ratio is always greater than some number r which is itself equal to or greater than 1.

Nothing is settled in case the test ratio is ultimately equal to 1. In this case other tests must be applied.

These theorems remain true if, not from the first, but after some particular term, say the kth, the test ratio has the values stated. For the sum of k terms is finite, hence the whole series will be convergent or divergent, according as the series beginning with the kth term is convergent or divergent.

The convergency test established in this article is known as the *test-ratio test.* It is sufficient for all the series treated in the remaining portion of this chapter and the chapter following. In fact, the test-ratio test will answer most purposes of elementary mathematics; the cases in which the test fails, that is when the test ratio approaches 1 in the limit, form exceptional cases which can usually be avoided. There are a great many other convergency tests by

which the convergency of a series can be settled in doubtful cases. The theory of series forms a separate subject of study, to the development of which many famous mathematicians have devoted their best efforts.

168. Convergency of Special Series.

(*a*) The *exponential series*,*

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

where $n!$ (factorial n) stands for $1 \times 2 \times 3 \times \cdots \times n$.

Here $u_{n+1} = \frac{x^n}{n!}$, $u_n = \frac{x^{n-1}}{(n-1)!}$, hence the test ratio

$$\frac{u_{n+1}}{u_n} = \frac{x^n}{n!} \div \frac{x^{n-1}}{(n-1)!} = \frac{x}{n}.$$

If n is taken sufficiently large, $\frac{x}{n}$ will become and remain less than 1 no matter how large x may be, provided only that it is finite; hence the series is absolutely convergent for every finite value of x.

(*b*) The *cosine series*

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \pm \frac{x^{2n}}{(2n)!} \mp \cdots$$

Here the general term is $\pm \frac{x^{2n}}{(2n)!}$, the preceding term $\mp \frac{x^{2n-2}}{(2n-2)!}$, hence the test ratio is

$$\frac{\pm x^{2n}}{(2n)!} \div \frac{\mp x^{2n-2}}{(2n-2)!} = \frac{-x^2}{2n(2n-1)}.$$

When n is taken sufficiently large, this ratio becomes and remains less than 1, therefore the series is absolutely convergent for all finite values of x.

(*c*) The *sine series*

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \pm \frac{x^{2n+1}}{(2n+1)!} \mp \cdots$$

* The reasons for the names given to the series in this article will appear shortly.

In this case the test ratio is

$$\frac{\pm x^{2n+1}}{(2n+1)!} \div \frac{\mp x^{2n-1}}{(2n-1)!} = \frac{-x^2}{2n(2n+1)}.$$

When n is taken sufficiently large this ratio becomes and remains less than unity, therefore the series is absolutely convergent for every finite value of x.

(*d*) The *logarithmic series*

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \pm \frac{x^n}{n} \mp \cdots$$

$u_n = \pm \frac{x^n}{n}$, $u_{n-1} = \mp \frac{x^{n-1}}{n-1}$, hence the test ratio is

$$\frac{\pm x^n}{n} \div \frac{\mp x^{n-1}}{n-1} = \frac{-(n-1)x}{n}.$$

$\frac{n-1}{n}$ is a proper fraction which approaches 1 as n approaches ∞.

The test ratio will therefore approach a number less than 1, equal to 1, or greater than 1, according as x is less than 1, equal to 1, or greater than 1.

The series is therefore absolutely convergent so long as x is less than 1. The test ratio fails to lead to any co clusion in the case x is equal to or greater than 1.

(*e*) The *binomial series*

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \cdots$$

The general term is

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r.$$

The preceding term is

$$\frac{n(n-1)(n-2)\ldots(n-r+2)}{(r-1)!}x^{r-1}.$$

Dividing the general term by the preceding term and canceling the common factors, the test ratio reduces to

$$\frac{n-r+1}{r}x = \left(\frac{n+1}{r} - 1\right)x.$$

As r approaches ∞, $\frac{n+1}{r}$ approaches 0; the test ratio therefore approaches $-x$ as its limit, and we may conclude that the series is absolutely convergent so long as x is less than 1.

Exercise 66

Write down the first six terms of each of the following series:

1. $\sum n^2$. 2. $\sum \frac{n}{n+1}$. 3. $\sum \frac{nx^n}{n^2+1}$. 4. $\sum \frac{(-1)^n 2^n x^n}{n!}$.

Write down the general term of each of the following series:

5. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots$

6. $\frac{x}{x+1}+\frac{x^2}{x+2}+\frac{x^3}{x+3}+\frac{x^4}{x+4}+\cdots$

7. $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots$

8. $\frac{x}{1}-\frac{x^2}{1+x}+\frac{x^3}{1+2x^2}-\frac{x^4}{1+3x^3}+\frac{x^5}{1+4x^4}-\cdots$

Show that the following series are absolutely convergent:

9. $1+\frac{2^2}{2}+\frac{3^2}{2^2}+\frac{4^2}{2^3}+\cdots+\frac{n^2}{2^{n-1}}+\cdots$

10. $1+\frac{2\cdot 2}{3}+\frac{3\cdot 2^2}{3^2}+\frac{4\cdot 2^3}{3^3}+\frac{5\cdot 2^4}{3^4}+\cdots+\frac{n\cdot 2^{n-1}}{3^{n-1}}+\cdots$

11. $2^2+\frac{3^2x}{2!}+\frac{4^2x^2}{3!}+\frac{5^2x^3}{4!}+\cdots+\frac{(n+1)^2x^{n-1}}{n!}+\cdots$

12. $1-\frac{3x^2}{2!}+\frac{5x^4}{4!}-\frac{7x^6}{6!}+\cdots+(-1)^n\frac{(2n+1)x^{2n}}{(2n)!}+\cdots$

Examine the following series as to convergency:

13. $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\cdots+\frac{n^2}{n!}+\cdots$

14. $1+\frac{x}{1}+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\cdots+\frac{x^n}{n^2}+\cdots$

Ans. Conv. for $x<1$.

15. $1+2!x+3!x^2+4!x^3+\cdots+n!x^{n-1}+\cdots$

Ans. Div.

16. $1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} + \cdots$

Ans. Conv. for $x < 1$, div. for $x > 1$.

17. $\frac{1}{2} - \frac{7}{6} + \frac{17}{12} - \frac{31}{20} + \frac{49}{30} - \frac{71}{42} + \cdots$

$$+ (-1)^{n-1}\left(\frac{n-1}{n} + \frac{n}{n+1}\right) + \cdots$$

(Suggestion. Find $S_2, S_3, S_4, \ldots, S_n = (-1)\frac{n-1}{n}$. From this it is seen that the series is oscillating.)

18. What is wrong with the following reasoning:

$$\text{Let } S' = 1 + \quad 3 + \quad 5 + \quad 7 + \quad 9 + \cdots$$
$$S'' = \quad 2 + \quad 4 + \quad 6 + \quad 8 + \quad 10 + \cdots$$

Adding $S' + S'' = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + \cdots$

$$2S' + 2S'' = 2 + 4 + 6 + 8 + 10 + \cdots = S'',$$

hence $2S' + S'' = 0$.

19. Show that the series

$$u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n-1} u_n + \cdots$$

where $u_1 > u_2 > u_3 > u_4 > \cdots > u_n$, and u_n approaches 0 as n approaches ∞, is certainly convergent.

169. The Number e. Consider the infinite series

$$1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} + \cdots \qquad (1)$$

The test ratio is $\frac{1}{n!} \div \frac{1}{(n-1)!} = \frac{1}{n}$, which is less than 1 for all values of n greater than 1, hence the series is convergent.

Let the sum of the series be denoted by e. The sum of the first three terms alone is 2.5, hence e is certainly greater than 2.5. We will show that e is less than 2.75.

$$\frac{1}{4!} = \frac{1}{3!\,4} < \frac{1}{3!\,3},$$

$$\frac{1}{5!} = \frac{1}{3!\,4 \cdot 5} < \frac{1}{3!\,3^2},$$

$$\frac{1}{6!} = \frac{1}{3!\,4 \cdot 5 \cdot 6} < \frac{1}{3!\,3^3}, \quad \text{etc.,}$$

hence

$$\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\cdots<\frac{1}{3!}+\frac{1}{3!\,3}+\frac{1}{3!\,3^2}+\frac{1}{3!\,3^3}+\cdots$$

The series on the right is a geometric series whose first term is $\frac{1}{3!}$ and whose ratio is $\frac{1}{3}$; its sum by Article 166 is equal to

$$\frac{1}{3!}\cdot\frac{1}{1-\frac{1}{3}}=\frac{1}{3!}\cdot\frac{3}{2}=\frac{1}{4}=0.25,$$

therefore
$$\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\cdots<0.25,$$

and since
$$1+\frac{1}{1}+\frac{1}{2!}=2.5,$$

therefore
$$e=1+\frac{1}{1}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{n!}+\cdots<2.75,$$

that is
$$2.5<e<2.75.$$

To determine e more exactly, we proceed as follows: Denote the terms of the series e by u_1, u_2, u_3, etc.

$$u_1 \qquad = 1.000\,000\,00$$

$$u_2=\frac{u_1}{1}=1.000\,000\,00$$

$$u_3=\frac{u_2}{2}=0.500\,000\,00$$

$$u_4=\frac{u_3}{3}=0.166\,666\,67$$

$$u_5=\frac{u_4}{4}=0.041\,666\,67$$

$$u_6=\frac{u_5}{5}=0.008\,333\,33$$

$$u_7=\frac{u_6}{6}=0.001\,388\,89$$

$$u_8=\frac{u_7}{7}=0.000\,198\,41$$

$$u_9=\frac{u_8}{8}=0.000\,024\,80$$

$$u_{10}=\frac{u_9}{9}=0.000\,002\,76$$

$$u_{11} = \frac{u_{10}}{10} = 0.000\ 000\ 28$$

$$u_{12} = \frac{u_{11}}{11} = 0.000\ 000\ 03$$

Adding $\qquad S_{12} \qquad = 2.718\ 281\ 84.$

This is the sum of the first twelve terms of the series for e. There is an error in the last figure, owing to the neglected part of each of the decimal fractions added, but this error cannot exceed 10×0.5 or 5 in the last decimal place. Besides this error there is the neglected portion of the series, which is $u_{13} + u_{14} + u_{15} + \cdots$.

Now
$$u_{13} = \frac{1}{13!}$$

$$u_{14} = \frac{1}{14!} < \frac{1}{13!\,13}$$

$$u_{15} = \frac{1}{15!} < \frac{1}{13!\,13^2}.$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

Adding $u_{13} + u_{14} + u_{15} + \cdots < \dfrac{1}{13!}\left(1 + \dfrac{1}{13} + \dfrac{1}{13^2} + \cdots\right)$

$$< \frac{1}{13!}\,\frac{1}{1 - \frac{1}{13}} = \frac{1}{12!\,12}.$$

From the computation above $u_{12} < 0.000\ 000\ 03$, hence the neglected portion of e is less than $\dfrac{u_{12}}{12} < 0.000\ 000\ 003$, that is, less than 3 in the ninth decimal place. Therefore

$$e = 2.718\ 281 \ldots \text{ correct to six places.}$$

170. The Exponential Series. By the binomial theorem

$$\left(1 + \frac{1}{n}\right)^n = 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \cdots$$

$$= 1 + 1 + \frac{\left(1 - \frac{1}{n}\right)}{2!} + \frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)}{3!} + \cdots. \qquad (1)$$

If n is greater than 1, each term of (1) is numerically equal to or less than the corresponding term of

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots \qquad (2)$$

hence the series (1) is convergent for every value of $n > 1$. As n approaches ∞, each term after the second approaches as its limit the corresponding term of (2), hence

the limit of $\left(1+\frac{1}{n}\right)^n$, as n approaches ∞, equals $e = 2.718 \cdots$. (3)

Next let us expand $\left(1+\frac{1}{n}\right)^{nx}$ by the binomial theorem.

$$\left(1+\frac{1}{n}\right)^{nx} = 1 + nx\,\frac{1}{n} + \frac{nx(nx-1)}{2!}\,\frac{1}{n^2} + \frac{nx(nx-1)(nx-2)}{3!}\,\frac{1}{n^3} + \cdots$$

$$= 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{2!} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{3!} + \cdots. \quad (4)$$

If n is greater than 1, each term of (4) is equal to or less than the corresponding term of the series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \quad (5)$$

which in Article 168 was shown to be convergent for every value of x, hence (4) is convergent for every value of x. As n approaches ∞, each term of (4), beginning with the third, approaches as its limit the corresponding term of (5). Hence, as n approaches ∞,

the limit of $\left(1+\frac{1}{n}\right)^{nx} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$ (6)

Finally, by the law of exponents

$$\left[\left(1+\frac{1}{n}\right)^n\right]^x = \left(1+\frac{1}{n}\right)^{nx} \quad (7)$$

no matter how large n and x may be. As n approaches ∞, the expression on the right approaches the series (6) as its limit, while the limit of the expression within the brackets on the left equals e, hence in the limit

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots. \quad (8)$$

The series (8) is known as the *exponential series* because it is the equivalent of the exponential function e^x.

The exponential series may be used to compute the number corresponding to any given natural logarithm. Assign to x any given value k, and let N be the number which is obtained by substituting k for x in the series (8). Then $e^k = N$, and since e is the base of the natural system of logarithms, Article 36, we have by the definition of a logarithm,

$$\log_e N = k,$$

that is, N is the number which has k for its natural logarithm.

171. The Logarithmic Series. Put $1+y=e^k$, then $k=\log_e(1+y)$, and

$$(1+y)^x = e^{kx} = e^{x\log_e(1+y)}.$$

By (8), Article 170,

$$(1+y)^x = e^{x\log_e(1+y)} = 1 + x\log_e(1+y) + \frac{[x\log_e(1+y)]^2}{2!} + \frac{[x\log_e(1+y)]^3}{3!} + \cdots$$

By the binomial theorem

$$(1+y)^x = 1 + xy + \frac{x(x-1)}{2!}y^2 + \frac{x(x-1)(x-2)}{3!}y^3 + \cdots$$

By the preceding article the first of these series is absolutely convergent, and so is the second provided $y < 1$, hence we may treat them like algebraic expressions containing a finite number of terms (Article 163). Equating the two series, subtracting 1 from each side and dividing out x, we obtain

$$\log_e(1+y)\left[1 + \frac{x}{2!}\log_e(1+y) + \frac{x^2}{3!}\log_e^2(1+y) + \cdots\right] = y + \frac{x-1}{2!}y^2 + \frac{(x-1)(x-2)}{3!}y^3 + \cdots$$

This equality holds for every value of x. As x approaches 0, the series within the brackets on the left approaches 1, and the series on the right approaches $y - \frac{y^2}{2} + \frac{y^3}{3} - \cdots$, hence in the limit

$$\log_e(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots + (-1)^{n-1}\frac{y^n}{n} + \cdots \quad (1)$$

Similarly, if y is negative, but numerically less than 1, we obtain

$$\log_e(1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \cdots - \frac{y^n}{n} - \cdots \quad (2)$$

The series (1) and (2), which differ only in the sign of y, are known as the *logarithmic series* because they are equivalent to the logarithms of $1 + y$ and $1 - y$ respectively.

By the aid of (1) and (2) the natural logarithm of any given number between 0 and 1 may be computed, but the actual computation of logarithms will be very much shortened by means of the method explained in the following article.

172. Calculation of Logarithms. The series (1) and (2) of the preceding article have been shown to be absolutely convergent (Article 168, (*d*)); we may therefore subtract the second from the first and obtain

$$\log_e(1+y) - \log_e(1-y) = \log_e \frac{1+y}{1-y} = 2\left(y + \frac{y^3}{3} + \frac{y^5}{5} + \frac{y^7}{7} + \cdots\right) \quad (1)$$

If in this series we put $y = \dfrac{1}{2n+1}$, we obtain

$$\log_e \frac{n+1}{n} = 2\left(\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \frac{1}{7(2n+1)^7} + \cdots\right)$$

or

$$\log_e(n+1) = \log_e n + 2\left(\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \frac{1}{7(2n+1)^7} + \cdots\right) \quad (2)$$

The series (2) is absolutely convergent so long as y is less than 1, that is, so long as n is greater than 0. If n is greater than 1 the series converges rapidly, so that but a few terms need be taken to obtain the first five or six decimal places of the number to which the series converges. The natural logarithm of any number may thus be readily computed provided we already know the logarithm of the next lower number. But the logarithm of 1 to any base is 0 (Article 26), hence the logarithm of 2 may be computed. Knowing $\log_e 2$ we may compute $\log_e 3$, thence $\log_e 4$, etc. Of course, only the logarithms of prime numbers need be computed by means of the series, for the logarithm of any composite number is equal to the sum of the logarithms of the factors of the number.

To compute the natural logarithm of 2 we put in (2) $n = 1$, thus

$$\log_e 2 = \log_e 1 + \left(\frac{2}{3} + \frac{2}{3 \cdot 3^3} + \frac{2}{5 \cdot 3^5} + \frac{2}{7 \cdot 3^7} + \cdots\right) \quad (3)$$

The actual calculation to five places of decimals may be conveniently arranged a follows. Denote the terms of the series in the parenthesis by u_1, u_2, u_3, etc., then

$$\log_e 1 \qquad\qquad = 0.000\,000\,0$$

$$\begin{array}{r|l}
3 & 2.000\,000\,0 \\
\hline
3^2 & 0.666\,666\,7 \div 1 = 0.666\,666\,7 = u_1 \\
\hline
3^2 & 0.074\,074\,1 \div 3 = 0.024\,691\,4 = u_2 \\
\hline
3^2 & 0.008\,230\,5 \div 5 = 0.001\,646\,1 = u_3 \\
\hline
3^2 & 0.000\,914\,5 \div 7 = 0.000\,130\,6 = u_4 \\
\hline
3^2 & 0.000\,101\,5 \div 9 = 0.000\,011\,3 = u_5 \\
\hline
3^2 & 0.000\,011\,3 \div 11 = 0.000\,001\,0 = u_6 \\
\hline
 & 0.000\,001\,3 \div 13 = \underline{0.000\,000\,1} = u_7
\end{array}$$

Adding

$$\log_e 1 + u_1 + u_2 + \cdots + u_7 = 0.693\,147\,2$$

There is an error in the last figure, due to the neglected parts of the fractions added, but this error cannot exceed 7×0.5 or 4 in the last decimal place. Besides, there is the neglected portion of the series, consisting of the terms

$$u_8 = \frac{2}{15 \cdot 3^{15}}$$

$$u_9 = \frac{2}{17 \cdot 3^{17}} < \frac{2}{15 \cdot 3^{15}} \cdot \frac{1}{3^2}$$

$$u_{10} = \frac{2}{19 \cdot 3^{19}} < \frac{2}{15 \cdot 3^{15}} \cdot \frac{1}{3^4}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Adding $u_8 + u_9 + u_{10} + \cdots < \dfrac{2}{15 \cdot 3^{15}}\left(1 + \dfrac{1}{3^2} + \dfrac{1}{3^4} + \cdots\right)$

The series within the brackets on the right is a geometric series whose ratio is $\frac{1}{3^2} = \frac{1}{9}$. The sum of this series is $\frac{1}{1 - \frac{1}{9}} = \frac{9}{8}$, hence the neglected portion of the series is less than $\frac{2}{15 \cdot 3^{15}} \cdot \frac{9}{8} = \frac{2}{120 \cdot 3^{13}}$.

From the computation above we already know that $\frac{2}{3^{13}} = 0.000\,001\,3$,

hence $\dfrac{2}{120 \cdot 3^{13}} = 0.000\,000\,01$. The error, to which the above sum 0.693 147 2 is subject, may therefore cause a difference of at most 1 in the sixth decimal place. It follows that $\log_e 2 = 0.693\,15\ldots$, correct to 5 places of decimals.

Similarly we obtain the following, each correct to 5 places of decimals:

$$\begin{aligned}
\log_e 3 &= \log_e 2 + 2\left(\frac{1}{5} + \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} + \frac{1}{7 \cdot 5^7} + \frac{1}{9 \cdot 5^9}\right) &&= 1.098\,61\ldots \\
\log_e 4 &= 2 \log_e 2 &&= 1.386\,29\ldots \\
\log_e 5 &= \log_e 4 + 2\left(\frac{1}{9} + \frac{1}{3 \cdot 9^3} + \frac{1}{5 \cdot 9^5} + \frac{1}{7 \cdot 9^7}\right) &&= 1.609\,44\ldots \\
\log_e 6 &= \log_e 2 + \log_e 3 &&= 1.791\,76\ldots \\
\log_e 7 &= \log_e 6 + 2\left(\frac{1}{13} + \frac{1}{3 \cdot 13^3} + \frac{1}{5 \cdot 13^5}\right) &&= 1.945\,91\ldots \\
\log_e 8 &= 3 \log_e 2 &&= 2.079\,44\ldots \\
\log_e 9 &= 2 \log_e 3 &&= 2.197\,22\ldots \\
\log_e 10 &= \log_e 2 + \log_e 5 &&= 2.302\,59\ldots
\end{aligned}$$

and so on.

It will be observed that the number of terms of the series, which need be computed to obtain the logarithm correct to a given number of decimal places, grows smaller as n grows larger. When n is 43 or more, the first term of the series suffices to give the first five places of the logarithm.

When the natural logarithm of a number is known, the common logarithm is easily obtained from it, for by Article 35

$$\log_{10} N = \frac{\log_e N}{\log_e 10}.$$

Now $\log_e 10$ was just found to be $2.302\,59\ldots$, hence

$$\log_{10} N = \frac{\log_e N}{2.302\,59\ldots} = 0.434\,29\ldots \log_e N \tag{4}$$

The number $0.434\,29\ldots$, more accurately $0.434\,294\,48\ldots$, is called the *modulus* * of the common system of logarithms. We have then the following simple rule.

* The base e, the modulus of the common system, and the natural logarithms of 2, 3 and 5, have been calculated to more than 250 places of decimals.

RULE: *To find the common logarithm of a number, multiply the corresponding natural logarithm by the modulus of the common system.*

Thus for the numbers from 2 to 10 inclusive we find the

Common Logarithms

$$\begin{aligned}
\log_{10} 2 &= 0.434\,294 \ldots \times 0.693\,15 \ldots = 0.301\,03 \ldots \\
\log_{10} 3 &= 0.434\,294 \ldots \times 1.098\,61 \ldots = 0.477\,12 \ldots \\
\log_{10} 4 &= 0.434\,294 \ldots \times 1.386\,29 \ldots = 0.602\,06 \ldots \\
\log_{10} 5 &= 0.434\,294 \ldots \times 1.609\,44 \ldots = 0.698\,97 \ldots \\
\log_{10} 6 &= 0.434\,294 \ldots \times 1.791\,76 \ldots = 0.778\,15 \ldots \\
\log_{10} 7 &= 0.434\,294 \ldots \times 1.945\,91 \ldots = 0.845\,10 \ldots \\
\log_{10} 8 &= 0.434\,294 \ldots \times 2.079\,44 \ldots = 0.903\,09 \ldots \\
\log_{10} 9 &= 0.434\,294 \ldots \times 2.197\,22 \ldots = 0.954\,24 \ldots \\
\log_{10} 10 &= 0.434\,294 \ldots \times 2.302\,59 \ldots = 1.000\,00
\end{aligned}$$

each correct to five places of decimals.

From (4) we have $\qquad \log_e N = 2.302\,59 \ldots \cdot \log_{10} N. \qquad (5)$

By means of (5) we can readily find any required natural logarithm from a table of common logarithms. It is therefore not very important to have separate tables of natural logarithms. Thus, if at any time we needed to know the natural logarithm of 237.3 we would look up the common logarithm of 237.3 and multiply the result by 2.30259 and obtain

$$\log_e 237.3 = 2.30259 \log_{10} 237.3 = 2.30259 \times 2.37530 = 5.46933 \ldots$$

173. Errors Resulting from the Use of Logarithms. In Article 44 were given certain rules governing the accuracy to be expected in the results obtained by the use of logarithmic tables containing a certain number of decimals. Some of these rules we are now able to verify.

Let N be any number obtained by logarithmic computation, and δ the error in the number due to the use of logarithms, so that the true result is $N + \delta$, where of course δ may be negative as well as positive. The difference between the logarithms of the true result and the result N is

$$\log_{10}(N + \delta) - \log_{10} N = \log_{10}\left(\frac{N+\delta}{N}\right) = \log_{10}\left(1 + \frac{\delta}{N}\right)$$

$$= \mu \log\left(1 + \frac{\delta}{N}\right), \text{ where } \mu = 0.43429 \cdots$$

(By Article 172, (4))

$$= \mu\left(\frac{\delta}{N} - \frac{\delta^2}{2N^2} + \frac{\delta^3}{3N^3} - \cdots\right)$$

(By Article 169, (1))

$$= \mu \frac{\delta}{N}, \text{ approximately,}$$

provided δ is small as compared with N.

This error $\mu \frac{\delta}{N}$ in the logarithm of N is the result of the errors in the logarithms as given in the tables.

In a four-place table the value of a unit in the last place is $0.0001 = \frac{1}{10^4}$, in a five-place table it is $0.00001 = \frac{1}{10^5}$, in an n-place table the value of the unit in the last place is $\frac{1}{10^n}$. The neglected part of any one logarithm does not exceed one-half of a unit in the last place; when several logarithms are added the positive and negative errors will tend to offset each other, but in special cases the errors may be accumulative. We will assume that the combined errors do not exceed a unit in the last place. This gives for an n-place table

$$\log_{10}(N + \delta) - \log_{10} N = \mu \frac{\delta}{N} \leqq \frac{1}{10^n},$$

from which

$$\delta \leqq \frac{N}{10^n \mu} = \frac{2.30259}{10^n} N.$$

Putting n successively equal to 3, 4, 5, 6, 7, we find:

In a 3-pl. table, $\delta \leqq 0.002\ 3\ N$, or less than $\frac{1}{4}$ of 1 % of N,

In a 4-pl. table, $\delta \leqq 0.000\ 23\ N$, or less than $\frac{1}{4}$ of $\frac{1}{10}$ % of N,

In a 5-pl. table, $\delta \leqq 0.000\ 023\ N$, or less than $\frac{1}{4}$ of $\frac{1}{100}$ % of N,

In a 6-pl. table, $\delta \leqq 0.000\ 002\ 3\ N$, or less than $\frac{1}{4}$ of $\frac{1}{1000}$ % of N,

In a 7-pl. table, $\delta \leqq 0.000\ 000\ 23\ N$, or less than $\frac{1}{4}$ of $\frac{1}{10000}$ % of N.

It follows that the third figure of a number found from a three-place table, the fourth figure of a number found from a four-place table, the fifth figure of a number found from a five-place table, etc., cannot be relied upon with certainty.

Exercise 67

1. Compute the natural logarithms of 11, 101, 257, each to four places of decimals. *Ans.* 2.3979, 4.6151, 5.5491.

2. From the results already obtained compute the natural logarithms of 12, 15, 0.05, $\frac{1}{3}$, each to four places of decimals.
Ans. 2.4849, 2.7081, −2.9957, −1.0986.

3. From the results of Problem 1 compute the common logarithms of 11, 101, 257, each to four places of decimals, and compare your results with those given in the table of common logarithms.

4. By means of a table of common logarithms, find the natural logarithms of 7, 341, 0.0473.
Ans. $\log_e 341 = 5.4848$, $\log_e 0.0473 = -3.0513$.

5. Prove that

$$\frac{1}{e} = \frac{2}{3!} + \frac{4}{5!} + \frac{6}{7!} + \cdots + \frac{2n}{(2n+1)!} + \cdots$$

6. Compute $\frac{1}{e}$ to five places of decimals, using the series in Problem 5. *Ans.* 0.36788 . . .

7. Show that the compound amount A on P dollars for t years at $r\%$, interest to be added to the principal as fast as it accrues, is

$$A = Pe^{rt}.$$

(Suggestion. Set up the expression for the compound amount when the period is the nth part of a year and find the limit which this expression approaches as n approaches ∞).

8. Sum the series $1 + \frac{1}{e} + \frac{1}{e^2} + \cdots + \frac{1}{e^n} + \cdots$

Ans. $\frac{e}{e-1}$.

174. Limiting Values of the Ratios $\frac{\sin x}{x}$, $\frac{\tan x}{x}$, x being the radian measure of the angle.

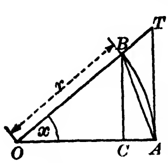


Fig. 177.

Let x be the radian measure of any angle less than $\pi/2$. With the vertex O as a center and any radius $OA = r$ describe an arc cutting the sides of the angle in A and B. Join A and B. From B draw a perpendicular to AO cutting AO in C. At A erect the perpendicular to AO cutting OB produced in T. Then

$$CB = r \sin x, \quad \text{arc}\, AB = rx, \quad AT = r \tan x.$$

The area of triangle $OAB = \dfrac{OA \times CB}{2} = \dfrac{r^2 \sin x}{2}$.

The area of sector $OAB = \dfrac{OA \times \text{arc}\, AB}{2} = \dfrac{r^2 x}{2}$.

The area of triangle $OAT = \dfrac{OA \times AT}{2} = \dfrac{r^2 \tan x}{2}$.

Also triangle $OAB <$ sector $OAB <$ triangle OAT,

that is,
$$\frac{r^2 \sin x}{2} < \frac{r^2 x}{2} < \frac{r^2 \tan x}{2},$$

from which
$$\sin x < x < \tan x \tag{1}$$

for every value of x less than $\dfrac{\pi}{2}$.

Dividing each term of (1) by the positive quantity $\sin x$,

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x},$$

hence
$$1 > \frac{\sin x}{x} > \frac{\cos x}{1}. \tag{2}$$

Now let x approach 0, then $\cos x$ approaches 1, therefore $\dfrac{\sin x}{x}$, which always lies between $\cos x$ and 1, must approach 1 also.

Similarly, if we divide (1) by $\tan x$

$$\cos x < \frac{x}{\tan x} < 1, \tag{3}$$

from which it is seen that $\dfrac{\tan x}{x}$ approaches 1 as x approaches 0.

Summing up,—

$$\left.\begin{array}{l} \text{As } x \text{ approaches } 0, \text{ the limit of } \dfrac{\sin x}{x} \text{ equals } 1. \\[2ex] \text{As } x \text{ approaches } 0, \text{ the limit of } \dfrac{\tan x}{x} \text{ equals } 1. \end{array}\right\} \tag{4}$$

Corollary.

$$\left.\begin{array}{l} \text{As } n \text{ approaches } \infty, \text{ the limit of } \dfrac{\sin(x/n)}{x/n} \text{ equals } 1. \\[2ex] \text{As } n \text{ approaches } \infty, \text{ the limit of } \dfrac{\tan(x/n)}{x/n} \text{ equals } 1. \end{array}\right\} \tag{5}$$

175. Limiting Value of $\cos^n \frac{x}{n}$ and $\left(\frac{\sin(x/n)}{x/n}\right)^n$ as n approaches ∞.

Put

$$y = \cos^n \frac{x}{n} = \left(1 - \sin^2 \frac{x}{n}\right)^{\frac{n}{2}},$$

then

$$\log y = \frac{n}{2} \log\left(1 - \sin^2 \frac{x}{n}\right)$$

$$= -\frac{n}{2}\left(\sin^2 \frac{x}{n} + \frac{1}{2}\sin^4 \frac{x}{n} + \frac{1}{3}\sin^6 \frac{x}{n} + \cdots\right), \text{ by Art. 171, (2).}$$

The series in the parenthesis is less than

$$\sin^2 \frac{x}{n} + \sin^4 \frac{x}{n} + \sin^6 \frac{x}{n} + \cdots,$$

which is a geometrical series, ratio $\sin^2 \frac{x}{n}$, whose sum is (Art. 166)

$$\frac{\sin^2(x/n)}{1 - \sin^2(x/n)} = \frac{\sin^2(x/n)}{\cos^2(x/n)} = \tan^2 \frac{x}{n},$$

hence, $\log y$ is numerically less than $\frac{n}{2}\tan^2 \frac{x}{n} = \frac{x}{2}\tan \frac{x}{n} \frac{\tan(x/n)}{x/n}$.

Now let n approach ∞, then $\frac{x}{n}$ approaches 0, $\tan \frac{x}{n}$ approaches 0, and $\frac{\tan(x/n)}{x/n}$ approaches 1 (Article 174, (5)), hence in the limit

$$\log y \text{ approaches } 0,$$

and consequently

$$y = \cos^n \frac{x}{n} \text{ approaches } 1.$$

Also from Article 174, (2),

$$1 > \frac{\sin(x/n)}{x/n} > \cos \frac{x}{n},$$

and therefore

$$1 > \left(\frac{\sin(x/n)}{x/n}\right)^n > \cos^n \frac{x}{n},$$

provided n is taken so large that $\frac{x}{n} < \frac{\pi}{2}$, that is, if n is taken suf-

ficiently large $\left(\frac{\sin(x/n)}{x/n}\right)^n$ lies always between 1 and $\cos^n \frac{x}{n}$, but the latter has just been shown to approach 1 as n approaches ∞, hence $\left(\frac{\sin(x/n)}{x/n}\right)^n$ also approaches 1. Summing up:

As n approaches ∞, the limit of $\cos^n \frac{x}{n}$ equals 1. (1)

As n approaches ∞, the limit of $\left(\frac{\sin(x/n)}{x/n}\right)^n$ equals 1. (2)

176. The Sine, Cosine and Tangent Series. In the equations (2) and (3), Article 159, namely,

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{2!}\cos^{n-2}\theta\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}\theta\sin^4\theta - \cdots$$

$$\sin n\theta = n\cos^{n-1}\theta\,\sin\theta - \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta\sin^3\theta + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}\cos^{n-5}\theta\sin^5\theta - \cdots$$

put $n\theta = x$, so that $\theta = \frac{x}{n}$, then

$$\cos x = \cos^n\frac{x}{n} - \frac{n(n-1)}{2!}\cos^{n-2}\frac{x}{n}\sin^2\frac{x}{n} + \frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}\frac{x}{n}\sin^4\frac{x}{n} - \cdots$$

$$\sin x = n\cos^{n-1}\frac{x}{n}\sin\frac{x}{n} - \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\frac{x}{n}\sin^3\frac{x}{n} + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}\cos^{n-5}\frac{x}{n}\sin^5\frac{x}{n} - \cdots$$

An examination of the test ratio shows that both series are absolutely convergent for every value of n. As n approaches ∞,

$$\cos^n\frac{x}{n} \text{ approaches } 1,$$

$$n\cos^{n-1}\frac{x}{n}\sin\frac{x}{n} = x\cos^{n-1}\frac{x}{n}\,\frac{\sin(x/n)}{x/n} \text{ approaches } x,$$

$$\frac{n(n-1)}{2!}\cos^{n-2}\frac{x}{n}\sin^2\frac{x}{n}=\left(1-\frac{1}{n}\right)\frac{x^2}{2!}\cos^{n-2}\frac{x}{n}\left(\frac{\sin(x/n)}{x/n}\right)^2 \text{ appr. } \frac{x^2}{2!},$$

and similarly

$$\frac{n(n-1)(n-2)}{3!}\cos^{n-3}\frac{x}{n}\sin^3\frac{x}{n} \text{ approaches } \frac{x^3}{3!},$$

$$\frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}\frac{x}{n}\sin^4\frac{x}{n} \text{ approaches } \frac{x^4}{4!}, \text{ etc.,}$$

Hence as n approaches ∞, the above series become

$$\cos x = 1 - \frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots \tag{1}$$

$$\sin x = x - \frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}+\cdots \tag{2}$$

These series being absolutely convergent (Article 168, (*b*), (*c*)), we may divide (2) by (1) and obtain

$$\tan x = x + \frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\frac{62x^9}{2835}+\cdots \tag{3}$$

The law of the exponents in each of the series (1), (2) and (3) is obvious, and so is the law of the coefficients in the series (1) and (2). The law of the coefficients in the series (3) is too complicated to be worked out by beginners.

177. Computation of Natural Function Tables. We know that the functions of any angle whatever may be expressed in terms of the functions of an angle less than 45°. Moreover, the sine and cosine of any angle between 30° and 45° may be expressed in terms of the sine and cosine of angles less than 30°, for by Article 113, (1),

$$\left.\begin{aligned}\sin x + \sin y &= 2\sin\frac{x+y}{2}\cos\frac{x-y}{2},\\ \cos x - \cos y &= -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}.\end{aligned}\right\} \tag{1}$$

If in (1) we put for x, $30° + \theta$, and for y, $30° - \theta$, and transpose the second terms on the left to the right-hand side of the equation, we obtain

$$\left.\begin{aligned}\sin(30°+\theta) &= 2\sin 30°\cos\theta - \sin(30°-\theta) = \cos\theta - \sin(30°-\theta),\\ \cos(30°+\theta) &= -2\sin 30°\sin\theta + \cos(30°-\theta) = -\sin\theta + \cos(30°-\theta),\end{aligned}\right\} \tag{2}$$

from which it appears that the functions of angles between 30° and 45° may be obtained from the functions of angles less than 30° by giving to θ successively the values from 0° to 15°. Thus, if $\theta = 5°$,

$$\sin(30° + \theta) = \sin 35° = \cos 5° - \sin 25°,$$
$$\cos(30° + \theta) = \cos 35° = -\sin 5° + \cos 25°.$$

A complete table of natural functions may therefore be readily constructed provided we know the functions of angles between 0° and 30°.

The sine and cosine of any angle less than 30° may be easily computed by means of the series in Article 176.

Suppose we wish to compute the sine and cosine of 10° correct to four places of decimals.

The radian measure of 10° is $x = \frac{\pi}{18} = 0.174\ 53\ldots$, and by means of logarithms we find

$$x^2 = 0.030\ 46\ldots, \qquad x^3 = 0.005\ 32\ldots,$$
$$x^4 = 0.000\ 93\ldots, \qquad x^5 = 0.000\ 16\ldots$$

Substituting these values in the series for the sine and cosine we have

$$\begin{array}{rlrl}
 & \cos x & & \sin x \\
 & 1.000\ 00 & x = & 0.174\ 53\ldots \\
-\frac{x^2}{2!} = & -0.015\ 23\ldots & -\frac{x^3}{3!} = & -0.000\ 89\ldots \\
+\frac{x^4}{4!} = & +0.000\ 04\ldots & +\frac{x^5}{5!} = & +0.000\ 00\ldots \\
\hline
\cos 10° = & 0.984\ 81\ldots & \sin 10° = & 0.173\ 64\ldots
\end{array}$$

In either case the error due to the neglected parts of the decimals added, cannot exceed a unit in the fifth place. In either case the error due to the neglected terms of the series is less than

$$\frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \cdots < \frac{x^6}{6!}(1 + x + x^2 + \cdots)$$

The series in parenthesis is a geometric series whose ratio is $x < 1$. Its sum is $\frac{1}{1-x}$, hence the neglected terms of either series add up to less than

$$\frac{x^6}{6\,!}\cdot\frac{1}{1-x} = \frac{(0.174\ 53\ldots)^6}{6\,!}\cdot\frac{1}{0.825\ 47\ldots} = 0.000\ 000\ 05\ldots$$

In neither case, therefore, can the combined errors affect the fourth decimal place, and we have

$$\cos 10^\circ = 0.9848\ldots,\quad \sin 10^\circ = 0.1736\ldots,$$

each correct to four places.

As a check we have

$$\cos^2 10^\circ + \sin^2 10^\circ = (0.9848)^2 + (0.1736)^2 = 1.0000.$$

It appears from the above computation that if the angle is less than 10° the sine and cosine are given correct to four places by the formulas

$$\cos x = 1 - \frac{x^2}{2\,!},\qquad \sin x = x - \frac{x^3}{3\,!}. \tag{3}$$

These approximation formulas are sufficiently important to be remembered.

We will next show how to compute a table of natural sines and cosines for intervals of $1'$.

By means of the sine and cosine series we first compute the sine and cosine of $1'$. We find

$$\sin 1' = 0.000\ 290\ 888\ 2\ldots,\quad \cos 1' = 0.999\ 999\ 957\ 7\ldots$$

If now we put in (1) $x = \theta + 1'$, $y = \theta - 1'$, we obtain

$$\begin{aligned} \sin(\theta + 1') &= 2\sin\theta\cos 1' - \sin(\theta - 1'),\\ \cos(\theta + 1') &= -2\sin\theta\sin 1' + \cos(\theta - 1'). \end{aligned} \tag{4}$$

Putting now $\theta = 1'$ we find

$$\begin{aligned} \sin 2' &= 2\sin 1'\cos 1' - \sin 0' = 0.000\ 581\ 776\ldots\\ \cos 2' &= -2\sin 1'\sin 1' + \cos 0' = 0.999\ 999\ 831\ldots \end{aligned}$$

Next put $\theta = 2'$, then

$$\begin{aligned} \sin 3' &= 2\sin 2'\cos 1' - \sin 1' = 0.000\ 872\ 665\ldots\\ \cos 3' &= -2\sin 2'\sin 1' + \cos 1' = 0.999\ 999\ 619\ldots \end{aligned}$$

Similarly, if $\theta = 3'$

$$\begin{aligned} \sin 4' &= 2\sin 3'\cos 1' - \sin 2' = 0.001\ 163\ 553\ldots\\ \cos 4' &= -2\sin 3'\sin 1' + \cos 2' = 0.999\ 999\ 322\ldots, \text{ etc.} \end{aligned}$$

To construct a table of sines and cosines for intervals of $10''$, we should first compute the sine and cosine of $10''$ and then make use of the formulas

$$\sin(\theta + 10'') = 2\sin\theta\cos 10'' - \sin(\theta - 10''),$$
$$\cos(\theta + 10'') = -2\sin\theta\sin 10'' + \cos(\theta - 10'').$$

178. Approximate Equality of Sine, Tangent and Radian Measure of very Small Angles. In Article 174 it was shown that the ratio of the sine to its angle expressed in radians, as well as the ratio of the tangent to its angle expressed in radians, approaches 1 as the angle approaches 0. This means that for very small angles the sine, the tangent and the angle expressed in radians are approximately equal. Thus, if we actually compute the sine and the tangent of $1''$ by means of the series in Article 176, and compare the results with the radian measure of $1''$, we shall find that the results agree to 15 places of decimals. The sine, tangent and radian measure of $1'$ agree to 11 places, and even when the angle is as large as $1°$ the sine, tangent and radian measure are equal so far as the first five places of decimals are concerned. It follows that when the angle is small, say $1°$ or less,* we may replace either or both the sine and the tangent by the angle expressed in radians, without affecting the first five places.

EXAMPLE 1. Find the smallest value of x that will satisfy the equation

$$2\sin x + 3x = 0.0513.$$

Solution.

$$3x = 0.0513 - 2\sin x$$
$$< 0.0513, \text{ that is, } x < 0.0171 \text{ (less than } 1°),$$

hence we may replace $\sin x$ by x. Then

$$2x + 3x = 5x = 0.0513,$$
$$x = 0.01026 \text{ radians} = 0° 35' 17''.$$

NOTE. The methods described in this chapter are not the methods that were used in calculating the tables now in use. The methods actually used were clumsy and laborious as compared with those we have studied. If the tables had to be calculated anew

* In fact, the first five places are the same up to $1° 59'$, that is, practically $2°$.

still more refined methods would be used, methods based on the calculus of finite differences, a branch of higher mathematics which cannot well be explained at this point.

By means of the differential calculus the sine and cosine series can be much more easily derived than has been done in this chapter. In the differential calculus all the series of Article 168 and many others are derived by means of a single theorem known as Taylor's Theorem.

Exercise 68

1. Calculate the sine and cosine of $5°$ correct to five places.
Ans. $\sin 5° = 0.08716$, $\cos 5° = 0.9962$.

2. Using the results of problem 1, calculate the tangent, cotangent, secant and cosecant of $5°$.
Ans. $\sec 5° = 1.0038$, $\csc 5° = 11.4737$.

3. By means of equations (4), Article 177, compute the sine and cosine of $5°$.

4. Compute the sine and cosine of $10''$ correct to 10 places.
Ans. $\sin 10'' = 0.000\,048\,481\,4$, $\cos 10'' = 0.999\,999\,998\,8$.

By means of the sine and cosine series verify the relations:

5. $\sin^2 x + \cos^2 x = 1$. 6. $\sin 2x = 2 \sin x \cos x$.

7. Find the first three terms of the series for $\sec x$.

$$\textit{Ans.}\quad \sec x = \frac{1}{\cos x} = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \cdots$$

8. An angle is to be corrected by an amount δ which is known to be less than $1°$ and is known to satisfy the equation

$$2\delta = 1.001599 - \cos\delta + \sin\delta.$$

Find δ expressed in minutes and seconds. *Ans.* $\delta = 5'\,30''$.

9. A straight rail AB, 1 mile long, whose extremities A and B are fixed, expands 1 inch, forming a curve ACB. Assuming the curve to be the arc of some circle, find the distance CD through which the middle point of the rail moved during the expansion.

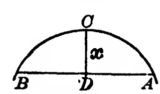


Fig. 178.

Ans. 12 ft. 10.14 in.

INDEX

(References refer to the pages.)